

TOPOLOGICAL TYPES OF 3-DIMENSIONAL SMALL COVERS

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ABSTRACT. In this paper we study the (equivariant) topological types of a class of 3-dimensional closed manifolds (i.e., 3-dimensional small covers), each of which admits a locally standard $(\mathbb{Z}_2)^3$ -action such that its orbit space is a simple convex 3-polytope. We introduce six equivariant operations on 3-dimensional small covers. These six operations are interesting because of their combinatorial natures. Then we show that each 3-dimensional small cover can be obtained from $\mathbb{R}P^3$ and $S^1 \times \mathbb{R}P^2$ with certain $(\mathbb{Z}_2)^3$ -actions under these six operations. As an application, we classify all 3-dimensional small covers up to $(\mathbb{Z}_2)^3$ -equivariant unoriented cobordism.

1. INTRODUCTION

Small covers are a class of particularly nicely behaving manifolds $M^n (n > 0)$, introduced by Davis and Januszkiewicz [4], each of which is an n -dimensional closed manifold with a locally standard $(\mathbb{Z}_2)^n$ -action such that its orbit space is a simple convex n -polytope P^n . There are strong links of small covers with combinatorics and polytopes. Davis and Januszkiewicz showed that small covers have very beautifully algebraic topology. For example, the equivariant cohomology of a small cover $\pi : M^n \rightarrow P^n$ is exactly isomorphic to the Stanley-Reisner face ring of P^n , and the mod 2 Betti numbers (b_0, b_1, \dots, b_n) of M^n agree with the h -vector (h_0, h_1, \dots, h_n) of P^n . In addition, they also showed that each small cover $\pi : M^n \rightarrow P^n$ determines a characteristic function λ (here we call it a $(\mathbb{Z}_2)^n$ -coloring) on P^n , defined by mapping all facets (i.e., $(n-1)$ -dimensional faces) of P^n to nonzero elements of $(\mathbb{Z}_2)^n$ such that n facets meeting at each vertex are mapped to n linearly independent elements, and conversely, up to equivariant homeomorphism, M^n can be reconstructed from the pair (P^n, λ) . More specifically, take a point x in the boundary ∂P^n , then there must be a l -dimensional face F^l of P^n such that x is in the relative interior of F^l , where $0 \leq l \leq n-1$. Since P^n is simple (i.e., the number of facets meeting at each vertex is exactly n), there are $n-l$ facets F_1, \dots, F_{n-l} such that $F^l = F_1 \cap \dots \cap F_{n-l}$. Let G_{F^l} denote the rank- $(n-l)$ subgroup of $(\mathbb{Z}_2)^n$ determined by $\lambda(F_1), \dots, \lambda(F_{n-l})$. Then we can define an equivalence relation \sim on the product bundle $P^n \times (\mathbb{Z}_2)^n$ as follows:

$$(x, g) \sim (y, h) \iff \begin{cases} x = y \text{ and } g = h & \text{if } x \text{ is contained in the interior of } P^n \\ x = y \text{ and } gh^{-1} \in G_{F^l} & \text{if } x \text{ is contained in the relative interior of} \\ & \text{some face } F^l \subset \partial P^n. \end{cases}$$

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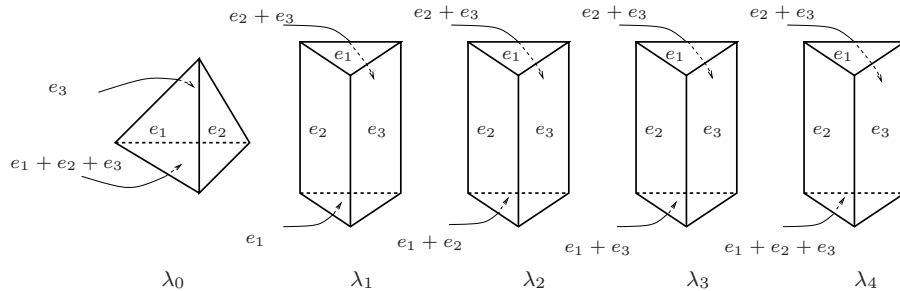
Furthermore, the quotient space $P^n \times (\mathbb{Z}_2)^n / \sim$ denoted by $M(P^n, \lambda)$ recovers M^n up to equivariant homeomorphism. Geometrically, $M(P^n, \lambda)$ is exactly obtained by gluing 2^n copies of P^n along their boundaries by using $(\mathbb{Z}_2)^n$ -coloring λ . This reconstruction of small covers provides a way of studying closed manifolds by using $(\mathbb{Z}_2)^n$ -colored polytopes.

In [8], Izmistiev studied a class of 3-dimensional small covers such that each λ of $(\mathbb{Z}_2)^3$ -colorings on their orbit spaces is 3-colorable (i.e., the image of λ contains only three linearly independent elements of $(\mathbb{Z}_2)^3$). Such a class of small covers are “pull-backs from the linear model” in the terminology of Davis and Januszkiewicz. Izmistiev obtained a classification result, saying that every such small cover can be produced from finitely many 3-dimensional tori with the canonical $(\mathbb{Z}_2)^3$ -action under the equivariant connected sum and the equivariant Dehn surgery.

In this paper, we shall consider all possible 3-dimensional small covers. Our objective is to determine the (equivariant) topological types of such a class of 3-dimensional manifolds. Four Color Theorem guarantees that each simple convex 3-polytope always admits a $(\mathbb{Z}_2)^3$ -coloring. Thus, by the reconstruction of small covers, simple convex 3-polytopes with $(\mathbb{Z}_2)^3$ -colorings can recover all 3-dimensional small covers, so all simple convex 3-polytopes will be involved in studying 3-dimensional small covers. Throughout this paper, we use the convention that if two simple convex polytopes P_1^3 and P_2^3 are combinatorially equivalent, then P_1^3 is identified with P_2^3 .

Let \mathcal{P} denote the set of all pairs (P^3, λ) where P^3 is a simple convex 3-polytope and λ is a $(\mathbb{Z}_2)^3$ -coloring on it, and let \mathcal{M} denote the set of all 3-dimensional small covers. Then, there is a one-to-one correspondence between \mathcal{P} and \mathcal{M} by mapping (P^3, λ) to $M(P^3, \lambda)$. There is a natural action of $\text{GL}(3, \mathbb{Z}_2)$ on \mathcal{P} , defined by the correspondence $(P^3, \lambda) \mapsto (P^3, \sigma \circ \lambda)$ where $\sigma \in \text{GL}(3, \mathbb{Z}_2)$. Obviously, this action is free, and it also induces an action of $\text{GL}(3, \mathbb{Z}_2)$ on \mathcal{M} by mapping $M(P^3, \lambda)$ to $M(P^3, \sigma \circ \lambda)$. Both $M(P^3, \lambda)$ and $M(P^3, \sigma \circ \lambda)$ are σ -equivariantly homeomorphic (cf [4]), so they are homeomorphic by forgetting their $(\mathbb{Z}_2)^3$ -actions. All elements of each equivalence class of $\mathcal{P}/\text{GL}(3, \mathbb{Z}_2)$ (resp. $\mathcal{M}/\text{GL}(3, \mathbb{Z}_2)$) are said to be $\text{GL}(3, \mathbb{Z}_2)$ -equivalent.

We shall first carry out our work on \mathcal{P} . We shall introduce six operations $\sharp^v, \sharp^e, \sharp^{eve}, \sharp, \sharp^\Delta, \sharp^\odot$ on \mathcal{P} . Then, under these six operations, up to $\text{GL}(3, \mathbb{Z}_2)$ -equivalence we find five basic pairs $(\Delta^3, \lambda_0), (P^3(3), \lambda_1), (P^3(3), \lambda_2), (P^3(3), \lambda_3), (P^3(3), \lambda_4)$ of \mathcal{P} where Δ^3 is a 3-simplex, $P^3(3)$ is a 3-sided prism, and $\lambda_i, i = 0, 1, \dots, 4$, are shown as in the following figure:



where $\{e_1, e_2, e_3\}$ is the standard basis of $(\mathbb{Z}_2)^3$. Then the combinatorial version of our main result is stated as follows.

Theorem 1.1. *Each pair (P^3, λ) in \mathcal{P} is an expression of $(\Delta^3, \sigma \circ \lambda_0)$, $(P^3(3), \sigma \circ \lambda_1)$, $(P^3(3), \sigma \circ \lambda_2)$, $(P^3(3), \sigma \circ \lambda_3)$, $(P^3(3), \sigma \circ \lambda_4)$, $\sigma \in \text{GL}(3, \mathbb{Z}_2)$, under six operations $\#^v$, $\#^e$, $\#^{eve}$, \natural , $\#^\Delta$, $\#^\odot$.*

By the reconstruction of small covers, six operations $\#^v, \#^e, \#^{eve}, \natural, \#^\Delta, \#^\odot$ on \mathcal{P} naturally correspond to six equivariant operations on \mathcal{M} , denoted by $\widetilde{\#^v}, \widetilde{\#^e}, \widetilde{\#^{eve}}, \widetilde{\natural}, \widetilde{\#^\Delta}, \widetilde{\#^\odot}$, respectively. These six operations can be understood very well because of their combinatorial natures. We shall see that $\widetilde{\#^v}$ is the equivariant connected sum, and $\widetilde{\natural}$ is the equivariant Dehn surgery, and other four operations $\widetilde{\#^e}, \widetilde{\#^{eve}}, \widetilde{\#^\Delta}, \widetilde{\#^\odot}$ can be understood as the generalized equivariant connected sums. By Theorem 1.1, $M(\Delta^3, \sigma \circ \lambda_0)$ and $M(P^3(3), \sigma \circ \lambda_i) (i = 1, \dots, 4)$, $\sigma \in \text{GL}(3, \mathbb{Z}_2)$, give all elementary generators of the algebraic system $\langle \mathcal{M}; \widetilde{\#^v}, \widetilde{\#^e}, \widetilde{\#^{eve}}, \widetilde{\natural}, \widetilde{\#^\Delta}, \widetilde{\#^\odot} \rangle$. On the other hand, we shall show that $M(\Delta^3, \lambda_0)$ is equivariantly homeomorphic to the $\mathbb{R}P^3$ with canonical linear $(\mathbb{Z}_2)^3$ -action, and $M(P^3(3), \lambda_i), i = 1, \dots, 4$, are equivariantly homeomorphic to the $S^1 \times \mathbb{R}P^2$ with four different $(\mathbb{Z}_2)^3$ -actions respectively. Then the topological version of our main result is stated as follows.

Theorem 1.2. *Each 3-dimensional small cover can be obtained from $\mathbb{R}P^3$ and $S^1 \times \mathbb{R}P^2$ with certain $(\mathbb{Z}_2)^3$ -actions by using six operations $\widetilde{\#^v}, \widetilde{\#^e}, \widetilde{\#^{eve}}, \widetilde{\natural}, \widetilde{\#^\Delta}, \widetilde{\#^\odot}$.*

Remark 1.1. Theorem 1.2 tells us how to obtain a 3-dimensional small cover from only two known small covers $\mathbb{R}P^3$ and $S^1 \times \mathbb{R}P^2$ with certain actions by using cut and paste strategies in the sense of six equivariant operations. This is an equivariant analogue of a well-known result ([10], [11], see also [9] and [16]) as follows: “Each closed 3-manifold can be obtained from a 3-sphere S^3 or a S^3 with one non-orientable bundle by using a finite number of Dehn surgeries”.

As an application, we study the $(\mathbb{Z}_2)^3$ -equivariant unoriented cobordism classification of all 3-dimensional small covers. Let $\widehat{\mathcal{M}}$ denote the set of $(\mathbb{Z}_2)^3$ -equivariant unoriented cobordism classes of all 3-dimensional small covers. Then $\widehat{\mathcal{M}}$ forms an abelian group under disjoint union, so it is also a vector space over \mathbb{Z}_2 .

Theorem 1.3. *$\widehat{\mathcal{M}}$ is generated by classes of $\mathbb{R}P^3$ and $S^1 \times \mathbb{R}P^2$ with certain $(\mathbb{Z}_2)^3$ -actions.*

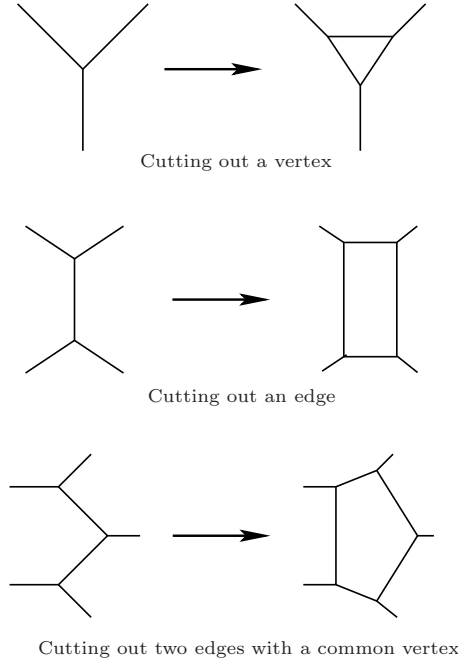
Remark 1.2. It should be pointed out that Theorem 1.3 is a direct corollary of main theorems in [14], but here we shall give it a different proof. Actually, the first author of this paper in [14] dealt with general closed 3-manifolds with effective $(\mathbb{Z}_2)^3$ -actions. Let \mathfrak{M}_3 be the \mathbb{Z}_2 -vector space consisting of $(\mathbb{Z}_2)^3$ -equivariant unoriented cobordism classes of all closed 3-manifolds with effective $(\mathbb{Z}_2)^3$ -actions. Then it was shown in [14] that \mathfrak{M}_3 can be generated by classes of $\mathbb{R}P^3$ and $S^1 \times \mathbb{R}P^2$ with certain $(\mathbb{Z}_2)^3$ -actions, and each class of \mathfrak{M}_3 contains a small cover as its representative. In particular, it was also shown in [14] that \mathfrak{M}_3 has dimension 13. Thus, $\widehat{\mathcal{M}}$ has dimension 13, too.

This paper is organized as follows. In Section 2 we establish the six operations on \mathcal{P} , and then we prove Theorem 1.1 in Section 3. In Section 4 we study elementary colored 3-polytopes and determine their equivariant topological types. Moreover, Theorem 1.2 is settled. In Section 5 we discuss how the corresponding six equivariant operations work on \mathcal{M} . As an application, we consider the $(\mathbb{Z}_2)^3$ -equivariant unoriented cobordism classification of all 3-dimensional small covers and prove Theorem 1.3 in Section 6.

2. OPERATIONS ON \mathcal{P}

The task of this section is to define six operations on \mathcal{P} . Throughout the remaining part of this paper, each nonzero element of $(\mathbb{Z}_2)^3$ is called a *color*, so $(\mathbb{Z}_2)^3$ contains seven colors.

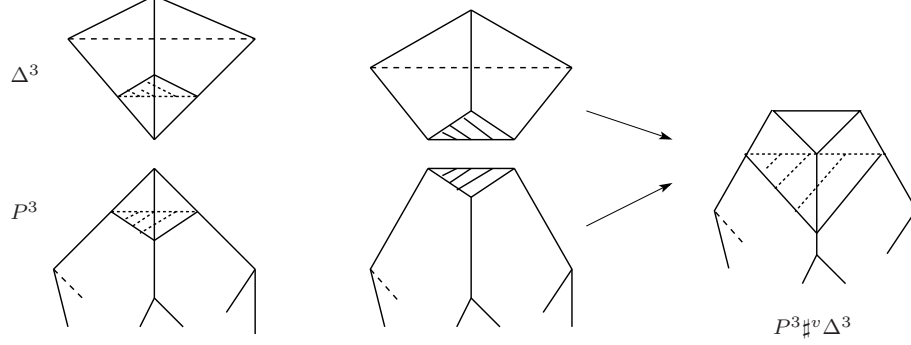
First, let us look at all simple uncolored 3-polytopes. It is well-known that any simple convex 3-polytope can be obtained from a 3-simplex by using three types of *excision* methods illustrated in the following figure: cutting out (i) a vertex; (ii) an edge; (iii) two edges with a common vertex. See Grünbaum's book [5, p.270].



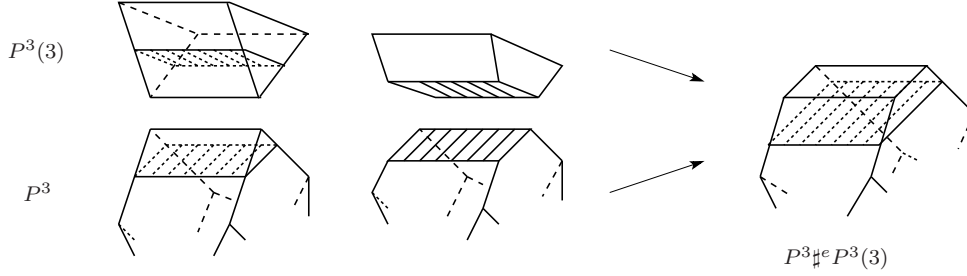
Since we shall carry out our study on colored polytopes and small covers, although these three types of excisions are very simple, they cannot directly work on colored polytopes and small covers because they will destroy the closedness of small covers. However, for our purpose we can interpret them as the “connected sum” operations with some standard simple 3-polytopes as follows.

2.1. Three operations \sharp^v , \sharp^e and \sharp^{eve} .

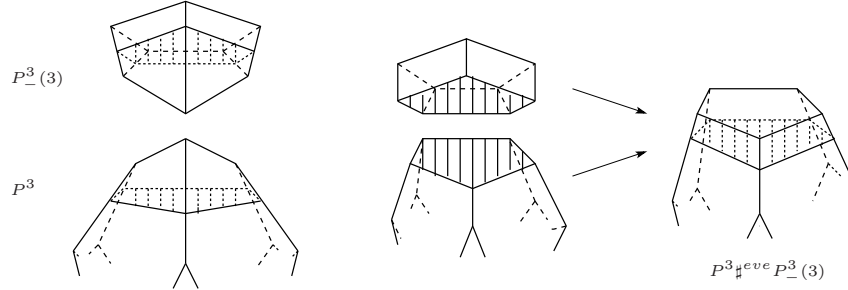
(I) The operation \sharp^v with a 3-simplex Δ^3



(II) The operation \sharp^e with a 3-sided prism $P^3(3)$



(III) The operation \sharp^{eve} with a truncated prism $P_-^3(3)$



Obviously, each of three operations is invertible as long as we don't perform the corresponding inverse operations of \sharp^v , \sharp^e , \sharp^{eve} on Δ^3 , $P^3(3)$, $P_-^3(3)$, respectively. Also, We always can do the operation \sharp^v between any two simple 3-polytopes. Since a 3-sided prism and a truncated prism can be obtained from a 3-simplex by using the operation \sharp^v , we have

Proposition 2.1. *Each simple 3-polytope can be obtained from a 3-simplex under three operations \sharp^v , \sharp^e and \sharp^{eve} .*

Definition 2.1. Let P^3 be a simple 3-polytope, and let F be a facet of P^3 . Then F is a ℓ -polygon where $\ell \geq 3$. If $\ell \leq 5$, then F is called a *small facet*; otherwise, it is called a *big facet*.

Also, for two edges with a common vertex in a simple 3-polytope P^3 , denoted by V_{eve} , there are at least four neighboring facets around V_{eve} . But it is easy to see that there

are exactly five neighboring facets around V_{eve} if V_{eve} is not in a triangular facet. Since V_{eve} is always associated with the operation \sharp^{eve} , throughout the rest of the paper, we use the *convention* that V_{eve} must be chosen in an m -polygonal facet with $m \geq 5$, so there are five neighboring facets around V_{eve} .

Suppose that P^3 is a simple 3-polytope but it is not a 3-simplex. Then we know by Proposition 2.1 that P^3 comes from applying one of the three types of cutting operations on some simple 3-polytope P'^3 such that the number of facets of P'^3 is one less than that of P^3 . In other words, there is a small facet F of P^3 such that P'^3 is obtained by compressing F into a point, or an edge or a V_{eve} in P^3 . In this case, we say that P^3 is *compressible at F* , and P'^3 is the *compression of P^3 at F* .

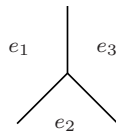
Corollary 2.2. *Suppose that P^3 is a simple 3-polytope other than a 3-simplex. Then P^3 is always compressible at some small facet.*

Now let us carry out our work on \mathcal{P} . We wish to know how the three operations \sharp^v , \sharp^e and \sharp^{eve} work on \mathcal{P} . To make our language more concise, first let us give some notions.

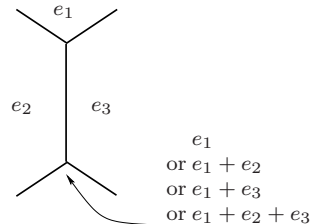
Definition 2.2 (Local colorings). Given a pair (P^3, λ) in \mathcal{P} . Let v be a vertex (or a 0-face) of P^3 . The colors of three facets meeting at v are said to be a *coloring of v* . Let e be an edge (or a 1-face) of P^3 . Then there must be four neighboring facets around e since P^3 is simple, and the colors of these four facets are said to be a *coloring of e* . Similarly, for a V_{eve} in P^3 , there are five neighboring facets around V_{eve} , and then the colors of those facets are said to be a *coloring of V_{eve}* .

Remark 2.1. By the definition of $(\mathbb{Z}_2)^3$ -colorings, the colors of neighboring facets around a vertex (resp. an edge and a V_{eve}) always can span the whole $(\mathbb{Z}_2)^3$. It is easy to see that up to $\text{GL}(3, \mathbb{Z}_2)$ -equivalence, a vertex admits a unique coloring, an edge admits four different kinds of colorings, and a V_{eve} admits 16 different kinds of colorings. We list them as follows:

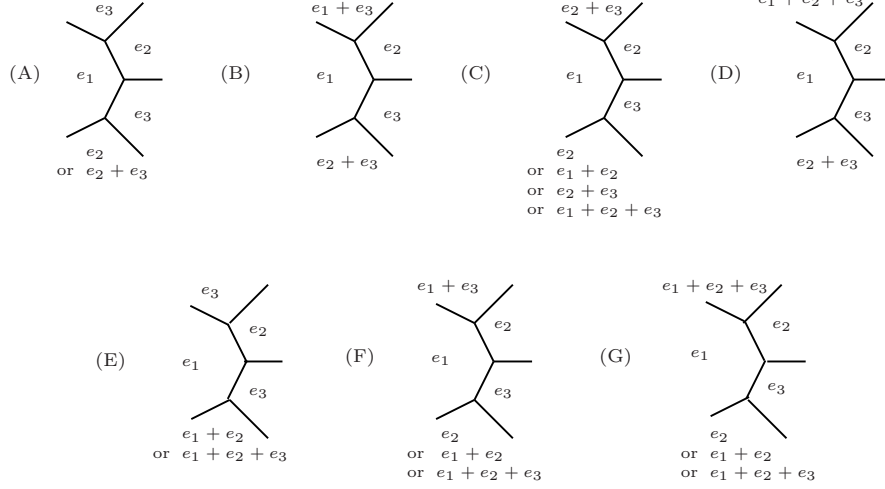
- (1) Colorings of a vertex and an edge



The coloring of a vertex



The colorings of an edge

(2) Colorings of a V_{eve}


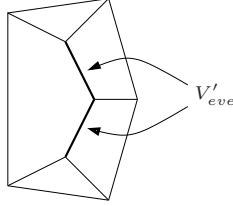
Definition 2.3. Given a pair (P^3, λ) in \mathcal{P} , and let F be a facet of P^3 . F is said to be *2-independent* if the colors of the neighboring facets around F span a 2-dimensional subspace of $(\mathbb{Z}_2)^3$. Similarly, F is said to be *3-independent* if the colors of the neighboring facets around F span the whole $(\mathbb{Z}_2)^3$.

With the above understood, let us look at how the three operations \sharp^v , \sharp^e and \sharp^{eve} work on \mathcal{P} .

Proposition 2.3. Up to $\text{GL}(3, \mathbb{Z}_2)$ -equivalence, the first two operations \sharp^v and \sharp^e can operate on any vertex and any edge in a colored simple 3-polytope, respectively, and the third operation \sharp^{eve} can operate on V_{eve} in a colored simple 3-polytope as long as the coloring of V_{eve} does not match any one of eight kinds of colorings shown in the figures (E)-(G) of Remark 2.1(2).

Proof. Let (P^3, λ) be a pair in \mathcal{P} . Choose a vertex v of P^3 , since v admits a unique coloring up to $\text{GL}(3, \mathbb{Z}_2)$ -equivalence, there is a pair (Δ^3, λ') such that some vertex in Δ^3 has the same coloring as v , so that we can do the operation \sharp^v between (P^3, λ) and (Δ^3, λ') . Choose an edge e of P^3 , then we know from Remark 2.1(1) that there are four kinds of colorings of e up to $\text{GL}(3, \mathbb{Z}_2)$ -equivalence, which agree with those colorings of an edge e' of $P^3(3)$, as shown in Section 1, where e' is not an edge of any triangle facet of $P^3(3)$. Thus, we can perform the operation \sharp^e on (P^3, λ) with some pair $(P^3(3), \lambda'')$. Choose a V_{eve} (i.e., two edges with a common vertex) in some facet F of P^3 (note that F is an m -polygon with $m \geq 5$ by our convention as before). We know from Remark 2.1(2) that there are 16 kinds of colorings of V_{eve} up to $\text{GL}(3, \mathbb{Z}_2)$ -equivalence. However, it is easy to see that the eight kinds of colorings shown in the figures (E)-(G) cannot be used as colorings of the neighboring facets around a pentagon in a simple 3-polytope by the definition of $(\mathbb{Z}_2)^3$ -colorings. This means that if V_{eve} has such a coloring, we can not perform the operation \sharp^{eve} on (P^3, λ) at V_{eve} . On the other hand, consider a V'_{eve}

in a truncated prism as shown in the following figure:



Obviously, V'_{eve} admits those eight kinds of colorings shown in the figures (A)-(D) of Remark 2.1(2), but it admits none of eight kinds of colorings shown in the figures (E)-(G) of Remark 2.1(2). Therefore, (P^3, λ) can do the operation \sharp^{eve} with a $P_-^3(3)$ with some $(\mathbb{Z}_2)^3$ -coloring if V_{eve} admits a coloring which matches one of the eight kinds of colorings shown in the figures (A)-(D) of Remark 2.1(2). \square

Remark 2.2. It should be pointed out that \sharp^v can always operate between any two pairs (P_1^3, λ_1) and (P_2^3, λ_2) in \mathcal{P} up to $\text{GL}(3, \mathbb{Z}_2)$ -equivalence. In fact, choose two vertices v_1 and v_2 in P_1^3 and P_2^3 respectively, then v_1 and v_2 have the same coloring up to $\text{GL}(3, \mathbb{Z}_2)$ -equivalence. Thus, by applying an automorphism $\sigma \in \text{GL}(3, \mathbb{Z}_2)$ to (P_1^3, λ_1) , we can change the coloring of v_1 into that of v_2 , so that we can do the operation \sharp^v between $(P_1^3, \sigma \circ \lambda_1)$ and (P_2^3, λ_2) . We shall see that \sharp^v exactly agrees with the equivariant connected sum of 3-dimensional small covers.

Similarly to the uncolored case, clearly we still cannot perform the corresponding inverse operations of $\sharp^v, \sharp^e, \sharp^{eve}$ on colored $\Delta^3, P^3(3), P_-^3(3)$, respectively. However, by Remark 2.1, it is easy to see that for any small facet F of a pair (P^3, λ) , if F is 2-independent, then (P^3, λ) cannot be compressed at F .

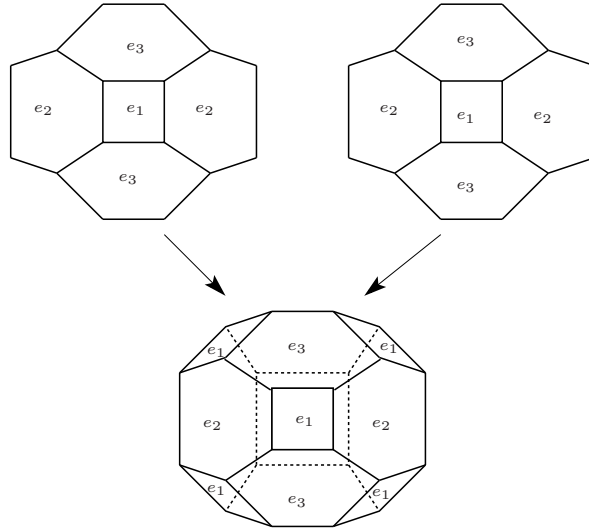
By Proposition 2.1, a natural question is whether each pair (P^3, λ) of \mathcal{P} can be produced only from a 3-simplex with $(\mathbb{Z}_2)^3$ -colorings in such three operations. However, generally the answer is *no*. For example, none of the four colorings on $P^3(3)$ as shown in Section 1 can be obtained from a 3-simplex with $(\mathbb{Z}_2)^3$ -colorings under three operations \sharp^v, \sharp^e and \sharp^{eve} . This is because each triangular facet in $P^3(3)$ with any one of those four colorings is 2-independent and it cannot be compressed into a point. More generally, we can further ask the following question:

(Q1): *Can any pair (P^3, λ) be produced by a 3-simplex, a prism and a truncated prism with $(\mathbb{Z}_2)^3$ -colorings under operations \sharp^v, \sharp^e and \sharp^{eve} ?*

Unfortunately, the answer is still *no*. Actually, generally it is possible that all the small facets are 2-independent, so we can not do the compression of (P^3, λ) at any of its small facets at all. This can be seen from the following example.

Example 2.1. Consider two copies of a square with four neighboring 6-polygons, we can glue them into a simple 3-polytope admitting a 3-colorable coloring, as shown in

the following figure:

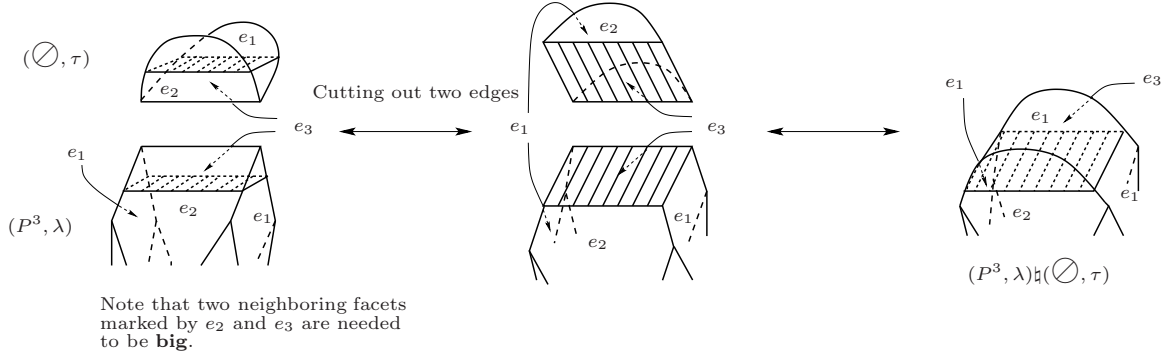


However, this 3-colorable example can not be compressed at any facet under operations \sharp^v , \sharp^e and \sharp^{eve} since each coloring on a 3-simplex (resp. a 3-sided prism, and a truncated prism) is not 3-colorable.

Remark 2.3. Generally, when a pair (P^3, λ) of \mathcal{P} is 3-colorable, a theorem of Izmistiev in [8] claims that (P^3, λ) can be obtained from a finite set of 3-colorable cubes by using the equivariant connected sum (i.e., the operation \sharp^v) and the equivariant Dehn surgery. The reason why his work was carried out very well is because the coloring of a 3-colorable polytope is unique up to $\text{GL}(3, \mathbb{Z}_2)$ -equivalence, while generally speaking, the set of all colorings given by more than three colors is quite complicated.

2.2. Operations \natural and \sharp^Δ on \mathcal{P} . According to the work of Izmistiev ([8]), we might need the fourth operation \natural on \mathcal{P} . This operation originally comes from the Dehn surgery on 3-manifolds rather than combinatorics. Based upon the topological meaning of Dehn surgery, Izmistiev gave it a combinatorial description by deleting a quarter of a cylinder with a subsequent gluing of a half-cylinder. Although this combinatorial description of the operation \natural works on \mathcal{P} very well, it doesn't meet the style of this paper, that is, it does not accord with the descriptions of other operations on \mathcal{P} in this paper. For this, we give another combinatorial description of this operation \natural , which is shown as

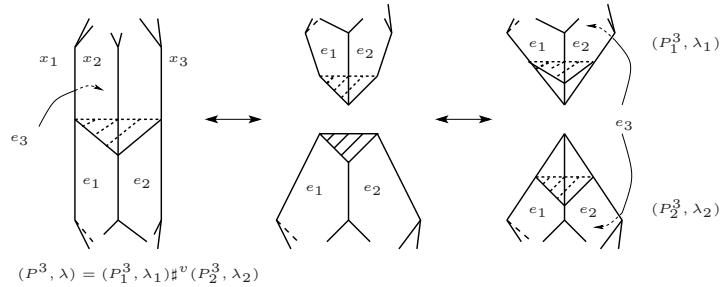
follows:



where \oslash is a quarter of a 3-ball, whose boundary consists of three 2-polygons, three edges and two vertices. Clearly \oslash is not a simple 3-polytope, but it still admits a $(\mathbb{Z}_2)^3$ -coloring. Note that \oslash is actually a nice manifold with corners ([3]). Obviously, the operation \natural is invertible. However, generally it may not be closed in \mathcal{P} because doing the operation \natural on a colored 3-polytope (P^3, λ) might destroy the 3-connectedness of the 1-skeleton of P^3 . In the 3-colorable case, Izmistiev showed that if \natural makes the 1-skeleton of the polytope not 3-connected, then one can find a connected sum somewhere else in the original polytope. In the general case, the argument of Izmistiev can be carried out to get a generalized result. The following is the combinatorial lemma proved by Izmistiev in [8] which will be used later in this paper.

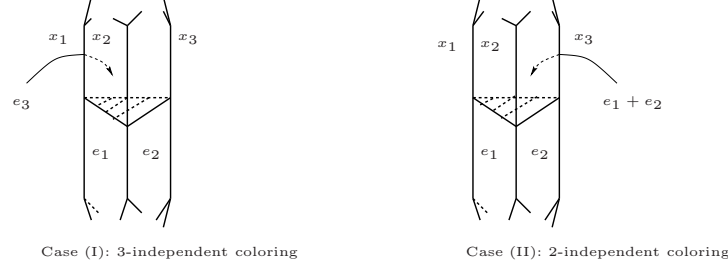
Lemma 2.4 ([8]). *If the 1-skeleton of a 3-polytope P is disconnected after cutting out three non-adjacent edges, then P can be written as $P = P_1 \#^v P_2$, where P_1, P_2 are 3-polytopes. In addition, when P is simple, so are P_1 and P_2 .*

Next, given a pair (P^3, λ) in \mathcal{P} , suppose that we can do an equivariant Dehn surgery on (P^3, λ) , but this operation destroys the 3-connectedness of the 1-skeleton Γ of P^3 . If λ is 3-colorable, Izmistiev gave a canonical method of finding three non-adjacent edges x_1, x_2, x_3 of P^3 such that $\Gamma \setminus \{x_1, x_2, x_3\}$ is disconnected (see [8] for the argument in detail). Then there are two 3-colorable pairs (P_1^3, λ_1) and (P_2^3, λ_2) such that $(P^3, \lambda) = (P_1^3, \lambda_1) \#^v (P_2^3, \lambda_2)$, as shown in the following figure:



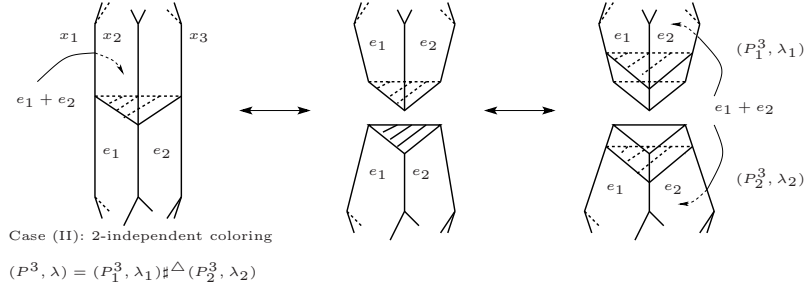
In the general case, we can still use the Izmistiev's method to find the required three non-adjacent edges x_1, x_2, x_3 such that $\Gamma \setminus \{x_1, x_2, x_3\}$ is disconnected, but there are *two possible colorings* up to $\text{GL}(3, \mathbb{Z}_2)$ -equivalence for three facets determined by x_1, x_2, x_3 ,

as shown in the following figure:



Obviously, the case (I) is the same as the 3-colorable case above, so there are two pairs (P_1^3, λ_1) and (P_2^3, λ_2) such that $(P^3, \lambda) = (P_1^3, \lambda_1) \#^v (P_2^3, \lambda_2)$. If the case (II) happens, then there still are two pairs (P_1^3, λ_1) and (P_2^3, λ_2) , but we need to introduce a new operation $\#^\Delta$, so that (P^3, λ) is equal to the sum of (P_1^3, λ_1) and (P_2^3, λ_2) under this new operation $\#^\Delta$.

The operation $\#^\Delta$ is defined as follows: first we cut out a triangular facet of (P_i^3, λ_i) , $i = 1, 2$, respectively, and then we glue them together along their triangular sections, as shown in the following figure:



Notice that the operation $\#^\Delta$ is invertible. It should be pointed out that the operation $\#^\Delta$ can also work in the case (I).

Combining the above argument, we have

Proposition 2.5. *Let (P^3, λ) be a pair in \mathcal{P} . Suppose that the 3-connectedness of P^3 is destroyed after doing an equivariant Dehn surgery \natural on (P^3, λ) . Then there are two pairs (P_1^3, λ_1) and (P_2^3, λ_2) in \mathcal{P} such that either $(P^3, \lambda) = (P_1^3, \lambda_1) \#^v (P_2^3, \lambda_2)$ or $(P^3, \lambda) = (P_1^3, \lambda_1) \#^\Delta (P_2^3, \lambda_2)$.*

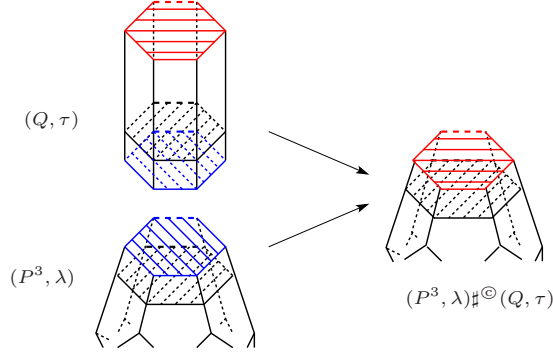
Next given a pair (P^3, λ) in \mathcal{P} and let F be a small facet. We wish to know

(Q2): Can (P^3, λ) always be compressed at F if F is 3-independent?

To answer this question (Q2), we need to introduce the following operation.

2.3. Operation $\#^\circ$ —Coloring change on \mathcal{P} . Now let us introduce the sixth operation $\#^\circ$ on \mathcal{P} . Given a pair (P^3, λ) in \mathcal{P} , we cannot avoid the occurrence of 2-independent facets in (P^3, λ) in general, but for our propose we can change their colorings. Let F be a 2-independent l -polygonal facet of (P^3, λ) . Then we can construct a l -sided prism $Q = F \times [0, 1]$, which naturally admits a coloring τ such that the coloring

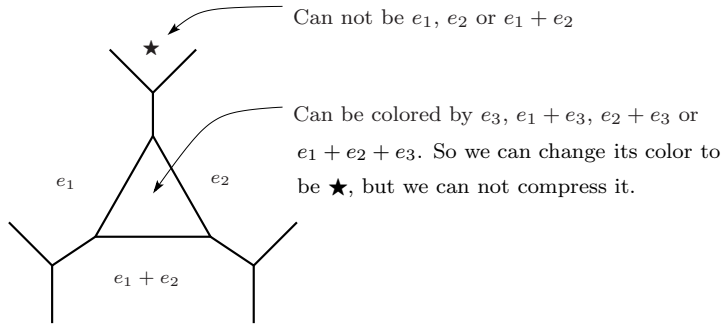
of the neighboring facets around the top facet (or bottom facet) is the same as that of F in (P^3, λ) . Since F is 2-independent, we can give two different colorings on the top facet and the bottom facet of Q , such that the bottom facet of Q has the same coloring as F . Then we can define an operation between (P^3, λ) and (Q, τ) as follows: cutting out the F of P^3 and the bottom facet of Q , and then gluing them together along sections, as shown in the following figure:



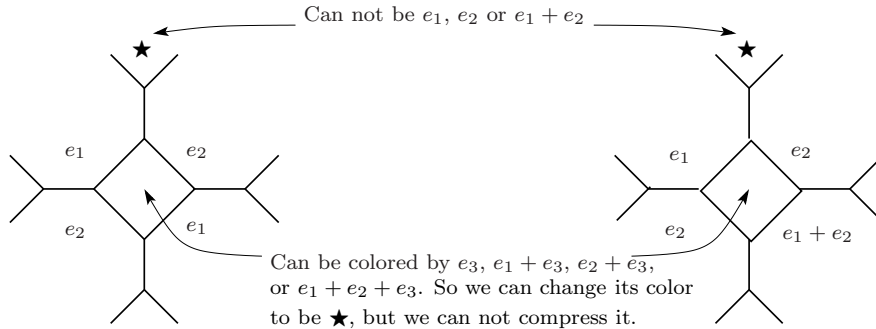
This operation exactly changes the coloring of F , so we also call it the *coloring change*, denoted by $\#^{\odot}$. Clearly, the operation $\#^{\odot}$ is invertible.

We shall mainly consider the coloring changes of 2-independent small facets, all possible cases (in the sense of $\text{GL}(3, \mathbb{Z}_2)$ -equivalence) of which are listed as follows:

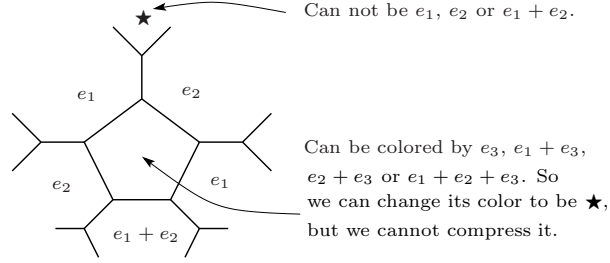
(a) triangular case



(b) rectangular case



(c) pentagonal case



Remark 2.4. As seen as above, when we do those six operations on \mathcal{P} , we need to cut out vertices, edges, V_{eve} 's, 2-independent triangular facets, 2-independent square facets, and 2-independent pentagonal facets, so that we can produce different kinds of sections on polytopes. By $S_v, S_e, S_{V_{eve}}, S_\Delta, S_\square$, and S_\star we denote those sections obtained by cutting out a vertex v , an edge e and a V_{eve} , a 2-independent triangular facet, a 2-independent square facet and a 2-independent pentagonal facet respectively. Also, the colorings of neighboring facets around $S_v, S_e, S_{V_{eve}}, S_\Delta, S_\square$, and S_\star are said to be the *colorings* of $S_v, S_e, S_{V_{eve}}, S_\Delta, S_\square$, and S_\star respectively. Obviously, these sections have the properties:

- (1) The colorings of $S_v, S_e, S_{V_{eve}}$ are all 3-independent. Up to $\text{GL}(3, \mathbb{Z}_2)$ -equivalence, S_v admits a unique coloring, S_e admits four different colorings, and $S_{V_{eve}}$ admits eight different colorings. The colorings of S_v and S_e agree with the colorings of a vertex and an edge respectively, see Remark 2.1(1). The colorings of $S_{V_{eve}}$ agree with the colorings shown in the figures (A)-(D) of Remark 2.1(2).
- (2) The colorings of $S_\Delta, S_\square, S_\star$ are all 2-independent. Up to $\text{GL}(3, \mathbb{Z}_2)$ -equivalence, S_Δ and S_\star admit a unique coloring, but S_\square admits two different colorings. These colorings agree with the colorings of around 2-independent small facets, as shown before Remark 2.4.

Notice that S_v and S_Δ are triangular sections, S_e and S_\square are rectangular sections, and $S_{V_{eve}}$ and S_\star are pentagonal sections.

Finally, let us discuss the question (Q2).

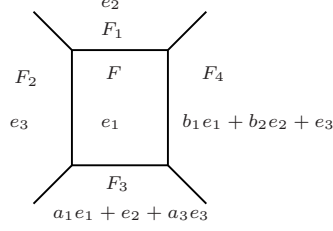
Proposition 2.6. *Let (P^3, λ) be a pair in \mathcal{P} and let F be a small facet of P^3 . Then (P^3, λ) is compressible at F if and only if F is 3-independent.*

Proof. Since we cannot perform the corresponding inverse operations of $\sharp^v, \sharp^e, \sharp^{eve}$ on colored $\Delta^3, P^3(3), P_-^3(3)$, respectively, we may assume that when F is a triangular (resp. rectangular, or pentagonal) facet, P^3 is not a 3-simplex (resp. a $P^3(3)$, or a $P_-^3(3)$). Obviously, if (P^3, λ) is compressible at F then F is 3-independent. Conversely, our argument proceeds as follows.

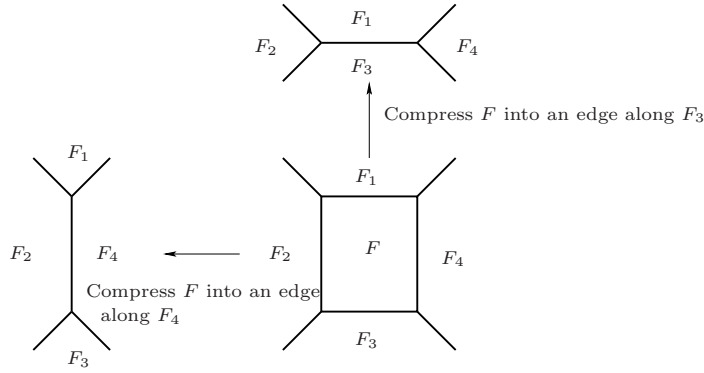
(1) Suppose that F is a 3-independent triangular facet. Then it is easy to see that (P^3, λ) is compressible at F .

(2) Suppose that F is a 3-independent rectangular facet. Then up to $\text{GL}(3, \mathbb{Z}_2)$ -equivalence, we may list all possible colorings of F and its four neighboring facets as

follows:

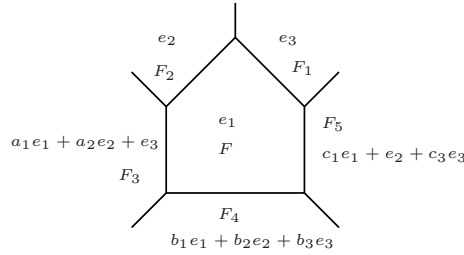


where $a_i, b_j \in \mathbb{Z}_2$ with $b_2a_3 = 0$, and at least one of a_1 and b_1 is nonzero. Obviously, if none of the four neighboring facets around F is triangular, then one can always compress F into an edge along F_3 (if $a_1 \neq 0$) or F_4 (if $b_1 \neq 0$), as shown in the following figure.



If there are triangular neighboring facets around F , then by Steinitz's Theorem the number of such triangular neighboring facets is at most 2. Since P^3 is not a $P^3(3)$ by our assumption, the number must be 1. With no loss assume that F_1 is a triangular facet. If F_1 is 3-independent, then one first compress F_1 into a point, so that F becomes a 3-independent triangular facet. Furthermore, one can compress F into a point. If F_1 is 2-independent and $a_1 = 1$, then one can compress F into an edge along F_3 ; if F_1 is 2-independent but $a_1 = 0$, then one can first change the coloring of F_1 into $e_1 + e_2$ by the operation \sharp^\odot , so that one can also compress F into an edge along F_3 .

(3) Suppose that F is a 3-independent pentagonal facet. Then up to $\text{GL}(3, \mathbb{Z}_2)$ -equivalence, all possible colorings of F and its five neighboring facets may be listed as follows:



where $a_i, b_j, c_k \in \mathbb{Z}_2$ with $\begin{cases} a_2b_3 + b_2 = 1 \\ b_3 + b_2c_3 = 1 \end{cases}$ and at least one of a_1, b_1 , and c_1 is nonzero.

An easy observation shows that if none of the five neighboring facets around F is

triangular, then there is always at least one neighboring facet F' around F so that one may compress F into a V_{eve} along F' . If there are triangular neighboring facets around F , then the Steinitz's Theorem makes sure that the number ℓ of such triangular neighboring facets is at most 2.

When the number ℓ is just 2, there are two possibilities. With no loss assume that either F_1 and F_3 or F_1 and F_4 are triangular. If F_1 and F_4 are triangular, then it is easy to check that at least one of F_1 and F_4 is 3-independent, so that one can compress the 3-independent triangular facet into a point and then F will become a rectangular facet. Thus, the problem is reduced to the case (2) above. If F_1 and F_3 are triangular, then there are two possible cases: either at least one of F_1 and F_3 is 3-independent or both F_1 and F_3 are 2-independent. If the former case happens, in a same way as above, one may reduce this case to the case (2). If the latter case happens, by changing the coloring of F_1 or F_3 via the operation $\#^\odot$ (if necessary), one may compress F into a V_{eve} along F_1 or F_3 .

When the number ℓ is just 1, with no loss assume that F_1 is triangular. If F_1 is 3-independent, then one may compress F_1 into a point, so that the case is reduced to the case (2). If F_1 is 2-independent, then an easy argument shows that, by adjusting the coloring of F_1 via the operation $\#^\odot$ (if necessary), one may always compress F into a V_{eve} along F_1 or F_3 or F_4 . \square

Remark 2.5. We see from the proof of Proposition 2.6 that if F is rectangular or pentagonal, then the manner of compressing F not only depends upon the colorings of the neighboring facets around F , but also the existence of the triangular neighboring facets around F . In general, the compression at F may need two possible steps: first compress the neighboring triangular facets around F , and then perform the compression of F . So after the compressions, F may be compressed into an edge or a point if F is rectangular, and a V_{eve} or an edge or a point if F is pentagonal.

3. PROOF OF THEOREM 1.1

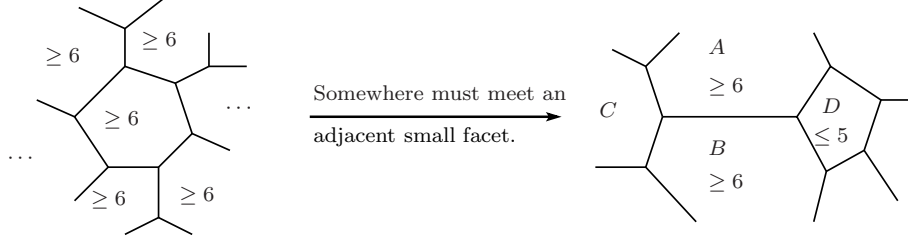
Let (P^3, λ) be a pair in \mathcal{P} . We shall finish the proof of Theorem 1.1 by using the descending induction on the number of facets of the simple 3-polytope P^3 . Without the loss of generality, assume that P^3 contains big facets.

First, by Proposition 2.6, we can compress all possible 3-independent small facets until we can not find them anymore. This does not increase the number of facets of P^3 . Let (P_c^3, λ_c) be the compression of (P^3, λ) under this step, and assume that (P_c^3, λ_c) still contains big facets. Then we divide our argument into two cases:

- (A) there are adjacent big facets in P_c^3 ;
- (B) there are no adjacent big facets in P_c^3 .

Case (A). Suppose that there are adjacent big facets in P_c^3 . Then there must be a pair of adjacent big facets such that there is an adjacent small facet as shown in the

following picture:



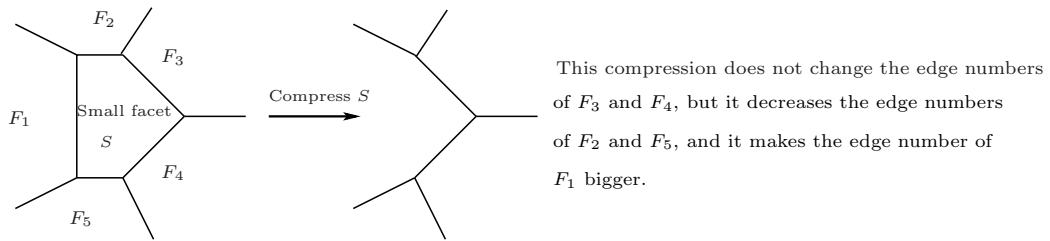
This is because the facets of P_c^3 are not all big according to the Euler characteristic of ∂P_c^3 . Next we try to do the equivariant Dehn surgery \natural on (P_c^3, λ_c) .

When C and D have the same coloring, we can do Dehn surgery \natural on (P_c^3, λ_c) , which would reduce the number of facets by one. If this operation doesn't destroy the 3-connectedness of 1-skeleton of P_c^3 , then we go on with our induction. Or else, by Proposition 2.5 we have that (P_c^3, λ_c) can be separated into two smaller pairs (P_1^3, λ_1) and (P_2^3, λ_2) such that either $(P_c^3, \lambda_c) = (P_1^3, \lambda_1) \#^v (P_2^3, \lambda_2)$ or $(P_c^3, \lambda_c) = (P_1^3, \lambda_1) \#^\Delta (P_2^3, \lambda_2)$. Then the problem is reduced to carrying out our inductions on (P_1^3, λ_1) and (P_2^3, λ_2) .

When C and D have different colorings, since the local coloring around D is 2-independent, by the operation $\#^\odot$ we can change the coloring of D to match the coloring of C . Then we can do the Dehn surgery operation, turning back to the above case.

The above procedure can always be carried out until we can not find adjacent big facets anywhere.

Case (B). If there are no adjacent big facets in P_c^3 , then any big facet is surrounded by 2-independent small facets. By the operation $\#^\odot$, we can change the coloring of a small facet, say F , then the adjacent small facets around F become 3-independent. Then we can compress them by using operations $\#^v$, $\#^e$ and $\#^{eve}$. We note that the edge number of the big facet will be reduced while we compress its neighboring triangular facets, and this number will be either reduced or unchanged while we compress its neighboring rectangular facets, but this number will be unchanged or reduced or becoming bigger while we compress its neighboring pentagonal facets, as shown in the following figure:



In particular, when we compress 3-independent pentagons, it is possible to produce new big facets. For example, if F_1 is a pentagon in the above figure, then it will become a big facet after compressing S . In addition, it is easy to see that compressing rectangles and pentagons may lead to the adjacency of big facets. If this happens, we can return

to the case (A) to do Dehn surgeries. Otherwise, by changing the colorings of small facets and compressing them, we can carry on our work to reduce the edge numbers of big facets.

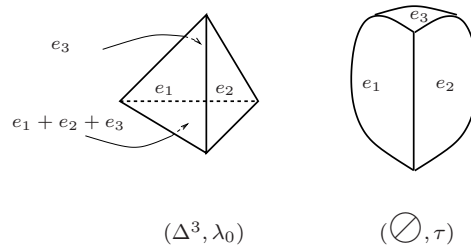
These alternate processes above can always end in finite steps since the number of facets of P^3 is finite. Note that with the help of \sharp^\odot , five operations \sharp^v , \sharp^e , \sharp^{eve} , \natural , \sharp^Δ can not only decrease the number of facets, but they can also decrease the edge numbers of big facets.

On the other hand, we see that the six operations themselves also involve some special colored blocks. Specifically, when doing three operations \sharp^v , \sharp^e and \sharp^{eve} , the colored Δ^3 , $P^3(3)$ and $P_-^3(3)$ are involved; when doing the operation \natural , the pair (\bigcirc, τ) is involved. As seen above, using the equivariant Dehn surgery \natural , we can avoid changing the colorings of big facets. So the special colored blocks involved in the operation \sharp^\odot are only those colored i -sided prisms $P^3(i)$ with top and bottom facets differently colored and being both 2-independent where $i = 3, 4, 5$.

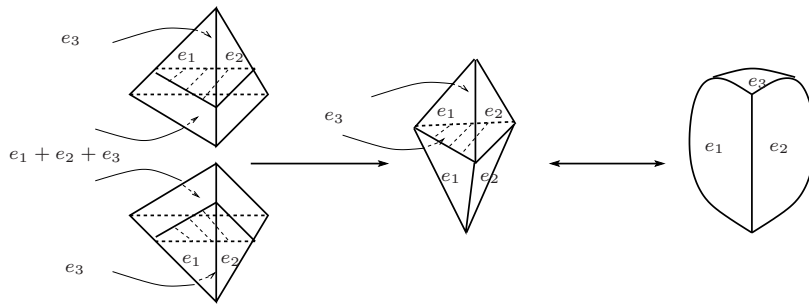
Combining Cases (A) and (B), we can always finish our induction by using the six types of operations until we reach one of those colored 3-polytopes Δ^3 , $P_-^3(3)$, and $P^3(i)$ ($i = 3, 4, 5$) above. Furthermore, by reversing the induction process, eventually (P^3, λ) can be described as an expression of those colored blocks Δ^3 , $P_-^3(3)$, $P^3(i)$ ($i = 3, 4, 5$), and \bigcirc under the six operations.

Next, to complete the proof, let us make a further analysis for those colored blocks Δ^3 , $P_-^3(3)$, $P^3(i)$ ($i = 3, 4, 5$), and \bigcirc .

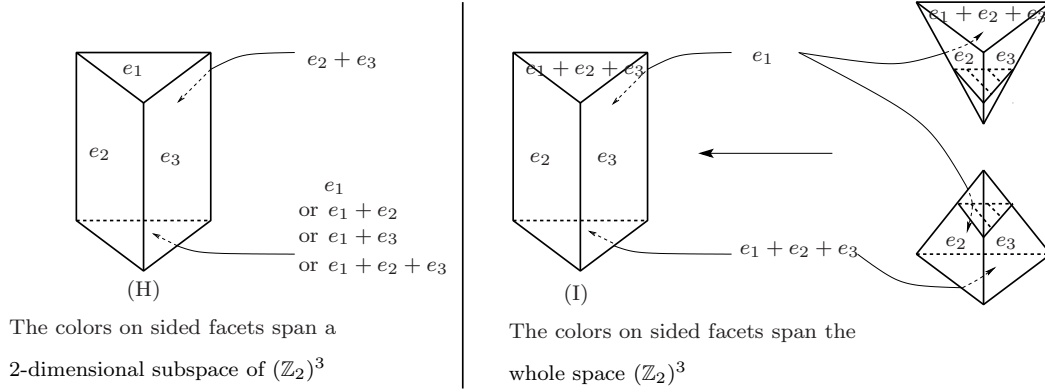
First it is easy to see that Δ^3 and \bigcirc admit a unique $(\mathbb{Z}_2)^3$ -coloring up to $\text{GL}(3, \mathbb{Z}_2)$ -equivalence, as shown in the following figure:



In particular, we have that $(\bigcirc, \tau) = (\Delta^3, \lambda_0) \sharp^\Delta (\Delta^3, \lambda_0)$, and this procedure is shown as follows:

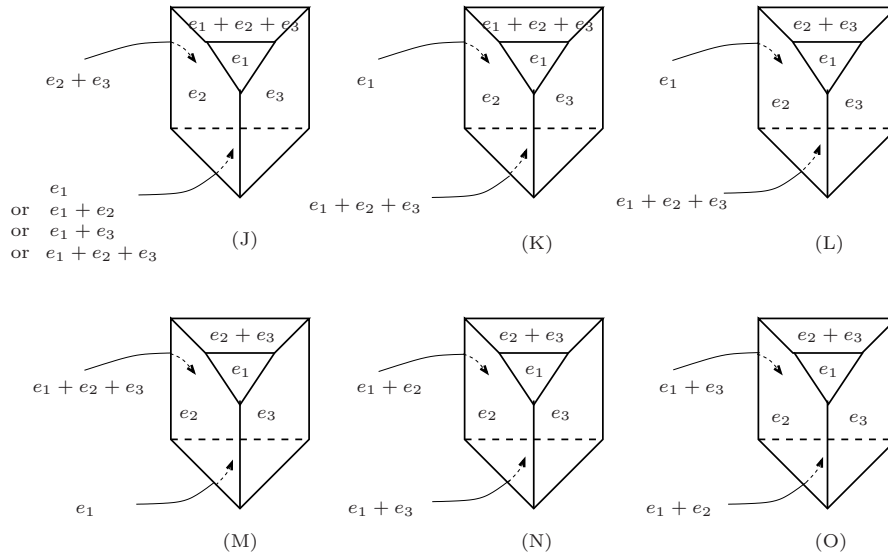


We know from [2, Theorem 3.1] that $P^3(3)$ admits five kinds of colorings up to $\text{GL}(3, \mathbb{Z}_2)$ -equivalence, which are listed as follows:



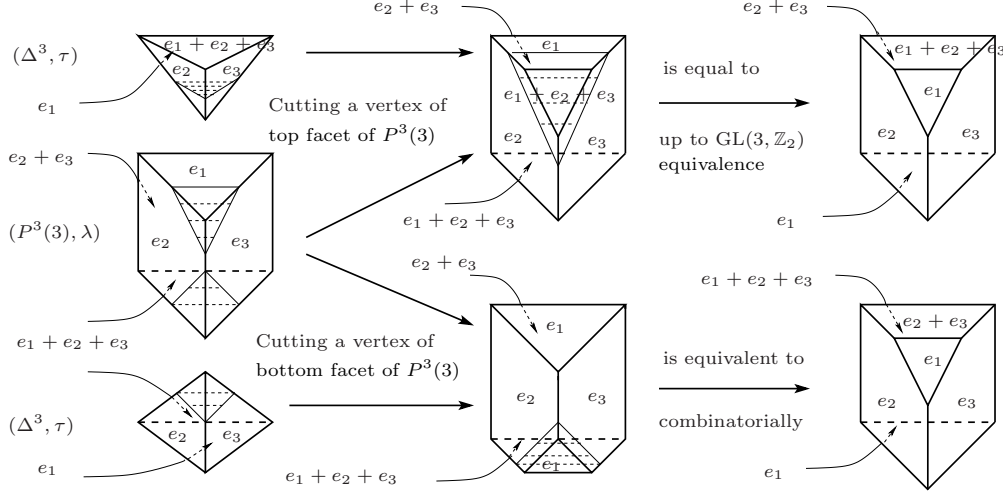
Obviously, the colored 3-sided prism on the right is the connected sum of two colored 3-simplices, as shown above. Now let us show that a colored $P_-^3(3)$ or $P^3(4)$ or $P^3(5)$ can be obtained from colored 3-simplices and 3-sided prisms via only two operations \sharp^v and \sharp^e .

- (a) From Figures (A)-(D) of Remark 2.1(2), we can obtain that $P_-^3(3)$ admits nine kinds of colorings up to $\text{GL}(3, \mathbb{Z}_2)$ -equivalence, as listed in the following figure:

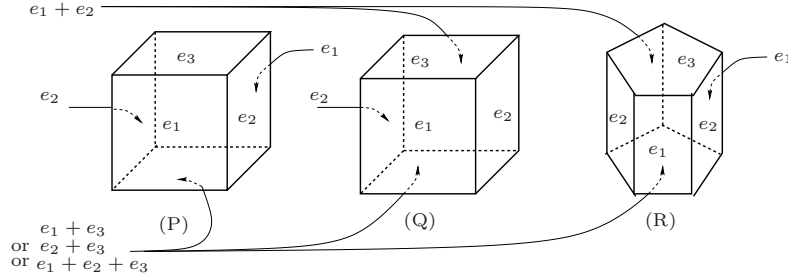


We claim that up to $\text{GL}(3, \mathbb{Z}_2)$ -equivalence, by doing the operation \sharp^v of colored $P^3(3)$'s with colored Δ^3 's, we can obtain the required nine kinds of colorings on $P_-^3(3)$. The argument is not difficult. See the following figure for two special cases, and all the other cases are similar to these two. Notice that the connected sums \sharp^v of a colored Δ^3 with some vertex of the top facet of a colored $P^3(3)$ and with some vertex of the bottom facet of a colored $P^3(3)$ respectively may

produce different colorings of $P_-^3(3)$.



- (b) Consider the colored prisms $P^3(4)$ and $P^3(5)$ with 2-independent top and bottom facets differently colored. It is easy to see that up to $\text{GL}(3, \mathbb{Z}_2)$ -equivalence and an automorphism h of $P^3(i)$, $P^3(4)$ admits six such colorings, and $P^3(5)$ admits three such colorings, where h is the automorphism of rotating facets on the side. We list them as follows:



Note that clearly h has no influence on the reconstruction of the above colored 3-polytopes up to equivariant homeomorphisms (cf [2]). Similarly to the case (a), an easy argument shows that each of colored 4-sided prisms shown in Figures (P) and (Q) is the sum of two colored 3-sided prisms under the operation $\#^e$, and each of colored 5-sided prisms shown in Figure (R) is the sum of a colored 3-sided prism and a colored 4-sided prism under the operation $\#^e$ so it is also the sum of three colored 3-sided prisms under the operation $\#^e$.

With all above arguments together, we see that, up to $\text{GL}(3, \mathbb{Z}_2)$ -equivalence there are only five elementary colored 3-polytopes as stated in Section 1, which can produce all colored 3-polytopes under the six operations. This completes the proof of Theorem 1.1. \square

Remark 3.1. For a colored m -sided prism $(P^3(m), \lambda)$, since its facets on the side are all squares, by considering 2-independence and 3-independence of square facets, we can use operations $\#^e$ and $\#^\odot$ alternately to compress facets on the side, so that $(P^3(m), \lambda)$

can be obtained from the colored $P^3(3)$ and $P^3(4)$. Since each colored 4-sided prism used in the operation \sharp^\odot above can be expressed as the sum of two colored $P^3(3)$'s under the operation \sharp^e by the proof of Theorem 1.1, we conclude that $(P^3(m), \lambda)$ is a sum of some colored $P^3(3)$'s under the operations \sharp^e and \sharp^\odot .

4. ELEMENTARY 3-DIMENSIONAL MANIFOLDS

The main task of this section is to determine those 3-dimensional small covers corresponding to (Δ^3, λ_0) and $(P^3(3), \lambda_i), i = 1, 2, 3, 4$, as stated in Section 1.

Recall that a small cover $\pi : M \rightarrow P$ is equivariantly homeomorphic to its reconstruction $M(P, \lambda)$ where the pair (P, λ) is determined by M . It is well-known (see [4] and [14]) that n -dimensional real projective space $\mathbb{R}P^n$ admits a canonical linear $(\mathbb{Z}_2)^n$ -action defined by

$$[x_0, x_1, \dots, x_n] \mapsto [x_0, g_1 x_1, \dots, g_n x_n]$$

where $(g_1, \dots, g_n) \in (\mathbb{Z}_2)^n$. This action fixes $n + 1$ fixed points $[0, \dots, 0, \underbrace{1, 0, \dots, 0}_i], i = 0, 1, \dots, n$, and its orbit space is homeomorphic to the image of the map $\Phi : \mathbb{R}P^n \rightarrow \mathbb{R}^{n+1}$ by

$$\Phi([x_0, x_1, \dots, x_n]) = \left(\frac{|x_0|}{\sum_{i=0}^n |x_i|}, \frac{|x_1|}{\sum_{i=0}^n |x_i|}, \dots, \frac{|x_n|}{\sum_{i=0}^n |x_i|} \right).$$

It is easy to see that the image of Φ is an n -dimensional simplex. A direct observation shows that the $n + 1$ facets of this n -simplex are colored by $e_1, \dots, e_n, e_1 + \dots + e_n$ respectively, where $\{e_1, \dots, e_n\}$ is the standard basis of $(\mathbb{Z}_2)^n$. This gives

Lemma 4.1. *$M(\Delta^3, \lambda_0)$ is equivariantly homeomorphic to the $\mathbb{R}P^3$ with a canonical linear $(\mathbb{Z}_2)^3$ -action.*

The product of $\mathbb{R}P^1 = S^1$ and $\mathbb{R}P^2$ with canonical linear actions gives a canonical $(\mathbb{Z}_2)^3$ -action (denoted by ϕ_1) on $S^1 \times \mathbb{R}P^2$, which has exactly six fixed points. Explicitly, this action on the product $S^1 \times \mathbb{R}P^2$ is defined by

$$\left((g_1, g_2, g_3), ((x_0, x_1), [y_0, y_1, y_2]) \right) \mapsto ((x_0, g_1 x_1), [y_0, g_2 y_1, g_3 y_2]).$$

The orbit space of this action on $S^1 \times \mathbb{R}P^2$ is the product of a 1-simplex and a 2-simplex, so it is just a 3-sided prism. It is also easy to see that the orbit space of this action admits the same coloring as $(P^3(3), \lambda_1)$. Thus we have

Lemma 4.2. *$M(P^3(3), \lambda_1)$ is equivariantly homeomorphic to the product $S^1 \times \mathbb{R}P^2$ with the canonical linear action ϕ_1 .*

Regard S^1 as the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$ in \mathbb{C} and $\mathbb{R}P^2$ as the projective plane $\mathbb{R}P(\mathbb{C} \oplus \mathbb{R}) = \{[v, w] \mid v \in \mathbb{C}, w \in \mathbb{R}\}$ in $\mathbb{C} \oplus \mathbb{R}$, we then construct three $(\mathbb{Z}_2)^3$ -actions ϕ_2, ϕ_3, ϕ_4 on $S^1 \times \mathbb{R}P^2$ as follows:

- (a) The action ϕ_2 on $S^1 \times \mathbb{R}P^2$ is defined by the following three commutative involutions

$$\begin{aligned} t_1 : (z, [v, w]) &\longmapsto (\bar{z}, [zv, w]) \\ t_2 : (z, [v, w]) &\longmapsto (z, [-\bar{z}\bar{v}, w]) \\ t_3 : (z, [v, w]) &\longmapsto (\bar{z}, [-zv, w]). \end{aligned}$$

- (b) The action ϕ_3 on $S^1 \times \mathbb{R}P^2$ is defined by the following three commutative involutions

$$\begin{aligned} t_1 : (z, [v, w]) &\longmapsto (\bar{z}, [zv, w]) \\ t_2 : (z, [v, w]) &\longmapsto (z, [\bar{z}\bar{v}, w]) \\ t_3 : (z, [v, w]) &\longmapsto (\bar{z}, [-zv, w]). \end{aligned}$$

- (c) The action ϕ_4 on $S^1 \times \mathbb{R}P^2$ is defined by the following three commutative involutions

$$\begin{aligned} t_1 : (z, [v, w]) &\longmapsto (\bar{z}, [\bar{z}v, w]) \\ t_2 : (z, [v, w]) &\longmapsto (z, [z\bar{v}, w]) \\ t_3 : (z, [v, w]) &\longmapsto (\bar{z}, [-z\bar{v}, w]). \end{aligned}$$

Note that the action ϕ_4 was first given in [14]. These three actions fix the same six points $(\pm 1, [1, 0])$, $(\pm 1, [\mathbf{i}, 0])$ and $(\pm 1, [0, 1])$, where $\mathbf{i} = \sqrt{-1}$.

Lemma 4.3. *$M(P^3(3), \lambda_i)$, $i = 2, 3, 4$, are equivariantly homeomorphic to $(S^1 \times \mathbb{R}P^2, \phi_i)$ respectively.*

Proof. First, let us show that each orbit space of the three actions is homeomorphic to a 3-sided prism $P^3(3)$. For $z \in S^1$ and $v \in \mathbb{C}$, write $z = e^{2\pi ti}$ and $v = re^{\theta \mathbf{i}}$ where $t \in [0, 1]$, $r \in \mathbb{R}_{\geq 0}$, and $\theta \in [0, 2\pi]$. Then we define the map $\Phi : S^1 \times \mathbb{R}P^2 \longrightarrow \mathbb{R}^5$ by

$$\Phi(z, [v, w]) = (x_1, x_2, x_3, x_4, x_5)$$

where

$$\begin{aligned} x_1 &= \frac{|\cos(2\pi t)|}{|\cos(2\pi t)| + |\sin(2\pi t)|}, & x_2 &= \frac{|\sin(2\pi t)|}{|\cos(2\pi t)| + |\sin(2\pi t)|}, \\ x_3 &= \frac{r|\cos(2\pi t + \theta)|}{r|\cos(2\pi t + \theta)| + r|\sin(2\pi t + \theta)| + |w|}, \\ x_4 &= \frac{r|\sin(2\pi t + \theta)|}{r|\cos(2\pi t + \theta)| + r|\sin(2\pi t + \theta)| + |w|}, \\ x_5 &= \frac{|w|}{r|\cos(2\pi t + \theta)| + r|\sin(2\pi t + \theta)| + |w|}. \end{aligned}$$

Notice that $\cos[2\pi(1-t) + \theta] = \cos(2\pi t - \theta)$ and $|\sin[2\pi(1-t) + \theta]| = |\sin(2\pi t - \theta)|$. Obviously, this map Φ is compatible with three actions ϕ_2, ϕ_3, ϕ_4 on $S^1 \times \mathbb{R}P^2$. In particular, we easily see that for each $t \in [0, 1]$, the image of Φ restricted to $\mathbb{R}P^2$ is a 2-simplex, which consists of all triples (x_3, x_4, x_5) . Also, the set $\{(x_1, x_2) | t \in [0, 1]\}$

forms a 1-simplex. Thus, the image of Φ is a 3-sided prism. Furthermore, it is easy to see that each orbit space of the three actions is homeomorphic to this 3-sided prism.

Next we show that the orbit space of the action ϕ_i admits the same coloring as $(P^3(3), \lambda_i)$. We shall only consider the case $i = 2$ because the arguments of other two cases are similar. Our strategy is to first determine the tangent representations at those fixed points and then to give the coloring on the orbit space by using algebraic duality.

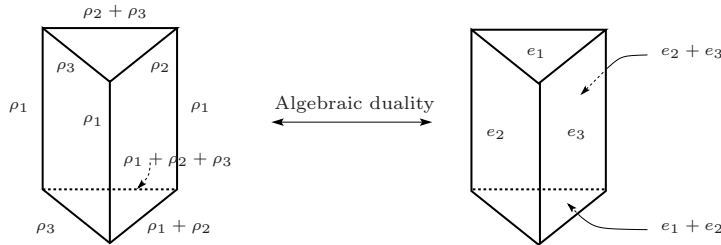
$\text{Hom}((\mathbb{Z}_2)^3, \mathbb{Z}_2)$, which consists all homomorphism from $(\mathbb{Z}_2)^3$ to \mathbb{Z}_2 , gives all irreducible representations of $(\mathbb{Z}_2)^3$, and forms an abelian group with addition given by $(\rho + \eta)(g) = \rho(g)\eta(g)$, where $g \in (\mathbb{Z}_2)^3$. The homomorphisms $\rho_j : g = (g_1, g_2, g_3) \mapsto g_j, j = 1, 2, 3$, form a basis of $\text{Hom}((\mathbb{Z}_2)^3, \mathbb{Z}_2)$. Now write $v = (v_1, v_2)$. When $z = -1$, the action ϕ_2 restricted to $\{-1\} \times \mathbb{R}P^2$ can be defined by the following way

$$\begin{aligned} (g, (-1, [v_1, v_2, w])) &\longmapsto (-1, [\rho_1(g)\rho_2(g)v_1, \rho_1(g)\rho_2(g)\rho_3(g)v_2, w]) \\ &= (-1, [\rho_3(g)v_1, v_2, \rho_1(g)\rho_2(g)\rho_3(g)w]) \\ &= (-1, [v_1, \rho_3(g)v_2, \rho_1(g)\rho_2(g)w]) \end{aligned}$$

and when $z = 1$, the action ϕ_2 restricted to $\{1\} \times \mathbb{R}P^2$ can be defined by the following way

$$\begin{aligned} (g, (1, [v_1, v_2, w])) &\longmapsto (1, [\rho_2(g)v_1, \rho_2(g)\rho_3(g)v_2, w]) \\ &= (1, [\rho_3(g)v_1, v_2, \rho_2(g)\rho_3(g)w]) \\ &= (1, [v_1, \rho_3(g)v_2, \rho_2(g)w]) \end{aligned}$$

Then we can read off the tangent representations at six fixed points, which determine a $\text{Hom}((\mathbb{Z}_2)^3, \mathbb{Z}_2)$ -coloring on 1-skeleton of the orbit space by GKM theory (see [6] and [12]), as shown in the following figure:



This $\text{Hom}((\mathbb{Z}_2)^3, \mathbb{Z}_2)$ -coloring is dual to the $(\mathbb{Z}_2)^3$ -coloring on the orbit space by $\rho_i(e_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$ (cf [13, Proposition 4.1]), so we can obtain the desired coloring, as shown in the above figure. \square

Although \bigcirc is not a 3-polytope, it is contractible, so we can apply the method of reconstruction of small covers to (\bigcirc, τ) to obtain a 3-manifold, denoted by $M(\bigcirc, \tau)$.

Lemma 4.4. *$M(\bigcirc, \tau)$ is equivariantly homeomorphic to the S^3 with the standard $(\mathbb{Z}_2)^3$ -action. Moreover, so is $M(\Delta^3, \lambda_0) \#^{\Delta^3} M(\Delta^3, \lambda_0)$.*

Proof. Consider the standard $(\mathbb{Z}_2)^3$ -action on S^3 by

$$(x_0, x_1, x_2, x_3) \mapsto (x_0, g_1 x_1, g_2 x_2, g_3 x_3).$$

Obviously, this action has two fixed points $(\pm 1, 0, 0, 0)$, and its orbit space is identified with \mathbb{O} . A direct observation shows that three 2-polygon faces of the orbit space are colored by e_1, e_2, e_3 , so this agrees with the coloring τ on \mathbb{O} . Since \mathbb{O} is contractible, any principal $(\mathbb{Z}_2)^3$ -bundle over \mathbb{O} is trivial. Furthermore, by the method of reconstruction it is easy to see that $M(\mathbb{O}, \tau)$ is equivariantly homeomorphic to the S^3 with the standard $(\mathbb{Z}_2)^3$ -action. \square

Remark 4.1. We easily see from [4, Theorem 3.1] that $M(\mathbb{O}, \tau)$ is not a small cover. In fact, any n -sphere S^n with $n > 1$ can not become a small cover. This is because its mod 2 Betti numbers $(1, 0, \dots, 0, 1)$ can not be used as the h -vector of any simple convex n -polytope. But S^1 is a small cover. Also, it is easy to see that both $M(\mathbb{O}, \tau)$ and $M(\mathbb{O}, \sigma \circ \tau)$ are σ -equivariantly homeomorphic, where $\sigma \in \text{GL}(3, \mathbb{Z})$.

By the reconstruction of small covers, together with Theorem 1.1 and Lemmas 4.1, 4.2, 4.3 and 4.4, we have completed the proof of Theorem 1.2. It remains to understand the geometrical meanings of corresponding six operations on \mathcal{M} .

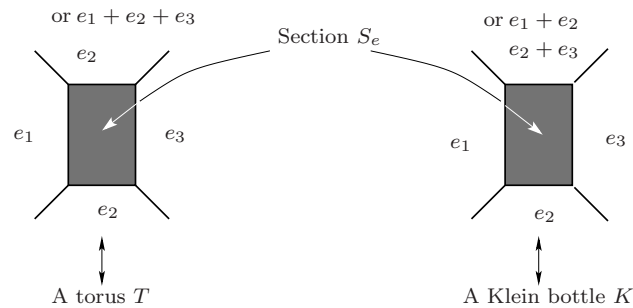
5. OPERATIONS ON \mathcal{M}

Now let us look at how corresponding six operations work on \mathcal{M} . In particular, this will tell us how to construct a small cover 3-manifold by using cut and paste strategies.

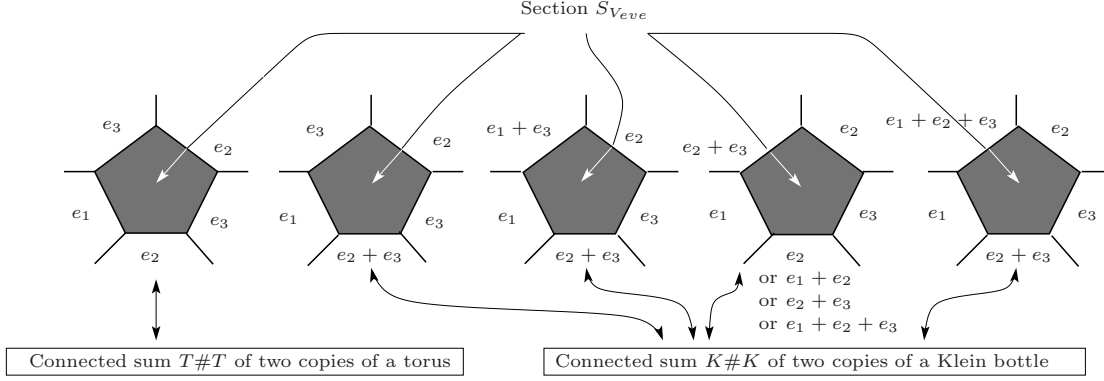
To understand six operations on \mathcal{M} , first let us study the corresponding geometrical meanings of sections $S_v, S_e, S_{V_{eve}}, S_\Delta, S_\square, S_\boxtimes$ in small covers. These sections actually correspond to some closed surfaces, which we list in the following lemma.

Lemma 5.1. *The corresponding geometrical meanings (up to homeomorphism) of sections $S_v, S_e, S_{V_{eve}}, S_\Delta, S_\square, S_\boxtimes$ in small covers are stated as follows:*

- (1) S_v corresponds to a 2-sphere S^2 ;
- (2) S_e corresponds to a 2-dimensional torus T or a Klein bottle K shown as follows:

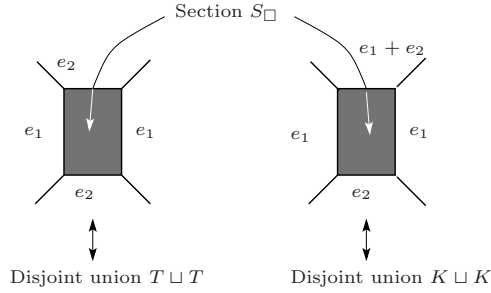


(3) $S_{V_{eve}}$ corresponds to a $T\#T$ or a $K\#K$ shown as follows:



(4) S_{Δ} corresponds to a disjoint union $\mathbb{R}P^2 \sqcup \mathbb{R}P^2$;

(5) S_{\square} corresponds to a $T \sqcup T$ or a $K \sqcup K$ shown as follows:



(6) S_{\boxtimes} corresponds to a disjoint union $(\mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2) \sqcup (\mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2)$,

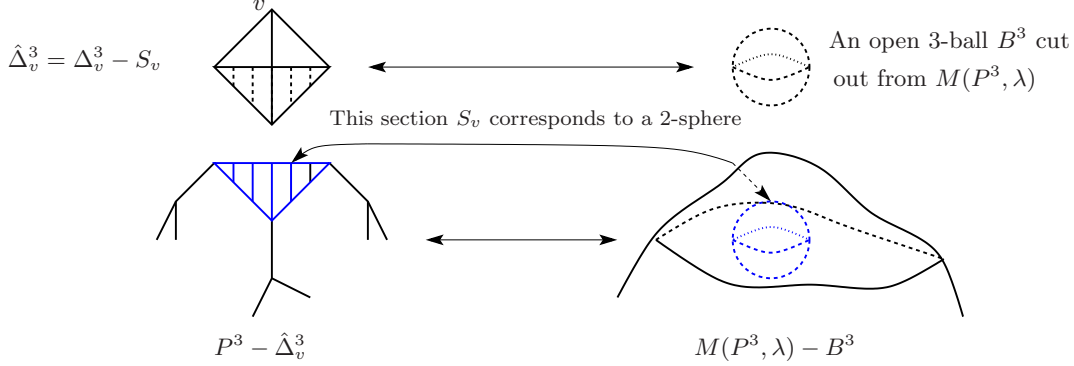
where $\#$ denotes the ordinary connected sum.

Proof. The argument is not quite difficult, and it is mainly based upon the reconstruction method of small covers. We would like to leave it to readers as an exercise. \square

Remark 5.1. Lemma 5.1 will play a beneficial role in understanding the six operations on \mathcal{M} . It should be pointed out that the corresponding closed surfaces of those sections are all not small covers in the sense of Davis-Januszkiewicz. Actually, for each such section S , its corresponding closed 2-manifold M^2 is the double covering space of a small cover over S . Also, if S is 2-independent then M^2 is disconnected, and if S is 3-independent then M^2 is connected.

5.1. Operation $\tilde{\#}^v$ on \mathcal{M} . This operation is actually the equivariant connected sum. By Lemma 5.1, cutting out a vertex v of a colored (P^3, λ) exactly corresponds to cutting out a $(\mathbb{Z}_2)^3$ -invariant open 3-ball which contains a fixed point of $M(P^3, \lambda)$ as shown in the following figure, so that the operation $\tilde{\#}^v$ on \mathcal{P} induces the equivariant connected

sum $\#^v$ on \mathcal{M} .



Now from the proof of Theorem 1.1 and by Lemmas 4.1, 4.2, 4.3 and 5.1, we have

Corollary 5.2. *The topological type of $M(P^3(3), \tau)$ is either $\mathbb{R}P^3 \# \mathbb{R}P^3$ or $S^1 \times \mathbb{R}P^2$. Furthermore, the topological type of $M(P_-^3(3), \tau)$ is either $\mathbb{R}P^3 \# \mathbb{R}P^3 \# \mathbb{R}P^3$ or $(S^1 \times \mathbb{R}P^2) \# \mathbb{R}P^3$.*

5.2. Operation $\#^e$ on \mathcal{M} . By Lemma 5.1, when we do the operation $\#^e$ on a $M(P^3, \lambda)$, we exactly cut out a $(\mathbb{Z}_2)^3$ -invariant open solid torus \widehat{T} (or a $(\mathbb{Z}_2)^3$ -invariant open solid Klein bottle \widehat{K}) from $M(P^3, \lambda)$, while we also need to cut out a same type of $(\mathbb{Z}_2)^3$ -invariant open solid torus (or a same type of $(\mathbb{Z}_2)^3$ -invariant open solid Klein bottle) from a $M(P^3(3), \tau)$. However, by Corollary 5.2 $M(P^3(3), \tau)$ has two different topological types: either $\mathbb{R}P^3 \# \mathbb{R}P^3$ or $S^1 \times \mathbb{R}P^2$. According to the colorings on $P^3(3)$, an easy argument shows that when the topological type of $M(P^3(3), \tau)$ is $\mathbb{R}P^3 \# \mathbb{R}P^3$, we can only cut out a $(\mathbb{Z}_2)^3$ -invariant open solid torus from $M(P^3(3), \tau)$, but when the topological type of $M(P^3(3), \tau)$ is $S^1 \times \mathbb{R}P^2$, we can not only cut out a $(\mathbb{Z}_2)^3$ -invariant open solid torus but also a $(\mathbb{Z}_2)^3$ -invariant open solid Klein bottle from $M(P^3(3), \tau)$. More precisely, up to $\text{GL}(3, \mathbb{Z}_2)$ -equivalence, when $\tau = \lambda_i, i = 1, 4$, we can only cut out a $(\mathbb{Z}_2)^3$ -invariant open solid torus from $M(P^3(3), \lambda_i)$ and when $\tau = \lambda_i, i = 2, 3$, we can only cut out a $(\mathbb{Z}_2)^3$ -invariant open solid Klein bottle from $M(P^3(3), \lambda_i)$. Thus, we have that if the topological type of $M(P^3(3), \tau)$ is $\mathbb{R}P^3 \# \mathbb{R}P^3$, then

$$M(P^3, \lambda) \#^e M(P^3(3), \tau) = (M(P^3, \lambda) \setminus \widehat{T}) \cup_T (M(P^3(3), \tau) \setminus \widehat{T})$$

and if the topological type of $M(P^3(3), \tau)$ is $S^1 \times \mathbb{R}P^2$, then

$$M(P^3, \lambda) \#^e M(P^3(3), \tau) = \begin{cases} (M(P^3, \lambda) \setminus \widehat{T}) \cup_T (M(P^3(3), \tau) \setminus \widehat{T}) & \text{if } \tau = \lambda_1, \lambda_4 \\ (M(P^3, \lambda) \setminus \widehat{K}) \cup_K (M(P^3(3), \tau) \setminus \widehat{K}) & \text{if } \tau = \lambda_2, \lambda_3. \end{cases}$$

5.3. Operation $\#^{eve}$ on \mathcal{M} . Similarly, by Lemma 5.1, when we do the operation $\#^{eve}$ on a $M(P^3, \lambda)$, we need to cut out a same type of $(\mathbb{Z}_2)^3$ -invariant $\widehat{T \# T}$ (or a same type of $(\mathbb{Z}_2)^3$ -invariant $\widehat{K \# K}$) from $M(P^3, \lambda)$ and $M(P_-^3(3), \tau)$ respectively, and then glue the remaining parts together along their boundaries, where $\widehat{T \# T}$ (resp. $\widehat{K \# K}$) denotes the interior of a 3-dimensional $(\mathbb{Z}_2)^3$ -manifold with boundary $T \# T$ (resp. $K \# K$). We know

from Corollary 5.2 that the topological type of $M(P_-^3(3), \tau)$ is either $\mathbb{R}P^3 \# \mathbb{R}P^3 \# \mathbb{R}P^3$ or $(S^1 \times \mathbb{R}P^2) \# \mathbb{R}P^3$. According to the colorings on $P_-^3(3)$, we see easily that if the topological type of $M(P_-^3(3), \tau)$ is $\mathbb{R}P^3 \# \mathbb{R}P^3 \# \mathbb{R}P^3$, then we can only cut out a $(\mathbb{Z}_2)^3$ -invariant $\widehat{T \# T}$ from $M(P_-^3(3), \tau)$, and if the topological type of $M(P_-^3(3), \tau)$ is $(S^1 \times \mathbb{R}P^2) \# \mathbb{R}P^3$, we can only cut out a $(\mathbb{Z}_2)^3$ -invariant $\widehat{K \# K}$ from $M(P_-^3(3), \tau)$. Therefore, we have that when the topological type of $M(P_-^3(3), \tau)$ is $\mathbb{R}P^3 \# \mathbb{R}P^3 \# \mathbb{R}P^3$,

$$M(P^3, \lambda) \#^{\widehat{eve}} M(P_-^3(3), \tau) = (M(P^3, \lambda) \setminus \widehat{T \# T}) \cup_{T \# T} (M(P_-^3(3), \tau) \setminus \widehat{T \# T})$$

and when the topological type of $M(P_-^3(3), \tau)$ is $(S^1 \times \mathbb{R}P^2) \# \mathbb{R}P^3$,

$$M(P^3, \lambda) \#^{\widehat{eve}} M(P_-^3(3), \tau) = (M(P^3, \lambda) \setminus \widehat{K \# K}) \cup_{K \# K} (M(P_-^3(3), \tau) \setminus \widehat{K \# K}).$$

5.4. Operation $\widetilde{\natural}$ on \mathcal{M} . Recall (cf [15] and [17]) that a $\frac{q}{p}$ -type Dehn surgery on a 3-manifold M^3 is as follows: removing a solid torus from M^3 and then sewing it back in M^3 such that the meridian goes to p times the longitude and q times the meridian, where $p, q \in \mathbb{Z}$.

Claim. The operation $\widetilde{\natural}$ on $M(P^3, \lambda)$ is exactly an equivariant $\frac{0}{1}$ -type Dehn surgery on $M(P^3, \lambda)$.

In fact, when we cut out an edge from (\bigcirc, τ) , the section is a 3-colorable square, so by Lemma 5.1 we exactly cut out a $(\mathbb{Z}_2)^3$ -invariant open solid torus from $M(\bigcirc, \tau)$. On the other hand, using the method of the reconstruction of small covers, the remaining part of the (\bigcirc, τ) can be reconstructed into a $(\mathbb{Z}_2)^3$ -invariant solid torus. So the operation $\widetilde{\natural}$ will remove a $(\mathbb{Z}_2)^3$ -invariant open solid torus N_1 from $M(P^3, \lambda)$ and glue back another $(\mathbb{Z}_2)^3$ -invariant solid torus N_2 come from $M(\bigcirc, \tau)$, mapping the meridian (longitude) of N_2 to the longitude (meridian) of N_1 . Notice that each edge in (P^3, λ) corresponds to a circle in $M(P^3, \lambda)$ by the reconstruction of small covers.

Therefore, the operation $\widetilde{\natural}$ on a $M(P^3, \lambda)$ up to $\text{GL}(3, \mathbb{Z}_2)$ -equivalence can be expressed as follows:

$$M(P^3, \lambda) \widetilde{\natural} M(\bigcirc, \tau) = (M(P^3, \lambda) \setminus \widehat{T}) \cup_T (M(\bigcirc, \tau) \setminus \widehat{T}).$$

5.5. Operation $\widetilde{\#^\Delta}$ on \mathcal{M} . When we do the operation $\widetilde{\#^\Delta}$ on two $M(P_1^3, \lambda_1)$ and $M(P_2^3, \lambda_2)$, since S_Δ corresponds to a disjoint union $\mathbb{R}P^2 \sqcup \mathbb{R}P^2$ by Lemma 5.1, we need to cut out a $(\mathbb{Z}_2)^3$ -invariant $\mathbb{R}P^2 \times (-1, 1)$ from each of both $M(P_1^3, \lambda_1)$ and $M(P_2^3, \lambda_2)$. Then we glue them together along their boundaries. Thus, we have

$$\begin{aligned} & M(P_1^3, \lambda_1) \widetilde{\#^\Delta} M(P_2^3, \lambda_2) \\ &= (M(P_1^3, \lambda_1) \setminus (\mathbb{R}P^2 \times (-1, 1))) \cup_{\mathbb{R}P^2 \sqcup \mathbb{R}P^2} (M(P_2^3, \lambda_2) \setminus (\mathbb{R}P^2 \times (-1, 1))). \end{aligned}$$

Notice that the two $(\mathbb{Z}_2)^3$ -invariant $\mathbb{R}P^2 \times (-1, 1)$ cut out from $M(P_1^3, \lambda_1)$ and $M(P_2^3, \lambda_2)$ may not always be equivariantly homeomorphic because we may cut out two triangular facets with different colorings from (P_1^3, λ_1) and (P_2^3, λ_2) .

5.6. Operation $\widehat{\#}^\circledast$ on \mathcal{M} . As we have seen, when we do the operation $\widehat{\#}^\circledast$ on \mathcal{P} , only 2-independent small facets are involved. Thus, when we do the operation $\widehat{\#}^\circledast$ on a $M(P^3, \lambda)$, by Lemma 5.1(4)-(6) we need to cut out a $(\mathbb{Z}_2)^3$ -invariant $\mathbb{R}P^2 \times (-1, 1)$, or a $(\mathbb{Z}_2)^3$ -invariant $T \times (-1, 1)$, or a $(\mathbb{Z}_2)^3$ -invariant $K \times (-1, 1)$, or a $(\mathbb{Z}_2)^3$ -invariant $(\mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2) \times (-1, 1)$ from $M(P^3, \lambda)$, and at the same time, up to $\text{GL}(3, \mathbb{Z}_2)$ -equivalence we also need to do same things on $M(P^3(i), \tau)$, $i = 3, 4, 5$, where the top facet and the bottom facet of each $(P^3(i), \tau)$ are colored by two different colors and the colorings of neighboring facets around them are 2-independent, as shown in Figures (H),(P)-(R) of Section 3, then gluing their corresponding boundaries together. When $i = 3$, by Lemmas 4.2 and 4.3, the topological type of $M(P^3(3), \tau)$ is exactly $S^1 \times \mathbb{R}P^2$. When $i = 4, 5$, we know from the proof of Theorem 1.1 that $M(P^3(4), \tau)$ is the sum of two $M(P^3(3), \eta_1)$ and $M(P^3(3), \eta_2)$ under $\widehat{\#}^e$, and $M(P^3(5), \tau)$ is the sum of a $M(P^3(3), \eta)$ and a $M(P^3(4), \kappa)$ under $\widehat{\#}^e$. However, this does not make clear what the topological types of $M(P^3(4), \tau)$ and $M(P^3(5), \tau)$ are. Next, we shall investigate their topological types.

It is well known (see [11]) that for any closed surface Σ , Σ -bundles over S^1 are classified by the mapping class group $\mathbf{MCG}^*(\Sigma)$. In particular,

- (I) when Σ is a torus T , $\mathbf{MCG}^*(T) \cong \text{SL}(2, \mathbb{Z}) = \text{Aut}(H_2(T, \mathbb{Z}))$.
- (II) when Σ is a Klein bottle K , $\mathbf{MCG}^*(K) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. In fact, if we think of K as $S^1 \times S^1 / (z_1, z_2) \sim (-z_1, \bar{z}_2)$, then elements in $\mathbf{MCG}^*(K)$ can be represented by $\{f_{\epsilon_1 \epsilon_2} \mid \epsilon_1 = \pm 1, \epsilon_2 = \pm 1\}$ where $f_{\epsilon_1 \epsilon_2}([z_1, z_2]) = ([z_1^{\epsilon_1}, z_2^{\epsilon_2}])$.

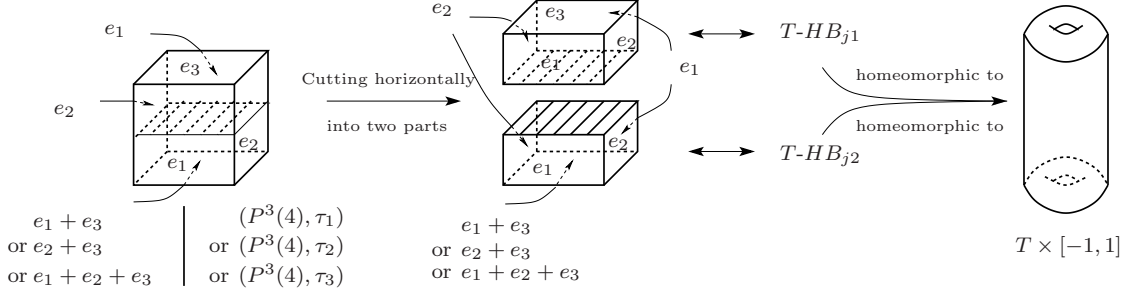
First let us look at the three colored 4-sided prisms shown in Figure (P) of Section 3, denoted by $(P^3(4), \tau_j)$, $j = 1, 2, 3$, respectively.

Lemma 5.3. *$M(P^3(4), \tau_j)$, $j = 1, 2, 3$, are equivariantly homeomorphic to three twisted T -bundles over S^1 with monodromy maps $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathbf{MCG}^*(T)$, respectively, where the $(\mathbb{Z}_2)^3$ -action on each twisted T -bundle over S^1 is induced by the $(\mathbb{Z}_2)^3$ -action ψ on $T \times [-1, 1]$ defined by the following three commutative involutions*

$$\begin{aligned} t_1 : (z_1, z_2, t) &\longmapsto (\bar{z}_1, z_2, t) \\ t_2 : (z_1, z_2, t) &\longmapsto (z_1, \bar{z}_2, t) \\ t_3 : (z_1, z_2, t) &\longmapsto (z_1, z_2, -t). \end{aligned}$$

Proof. By Lemma 5.1, any horizontal section of each $(P^3(4), \tau_j)$ corresponds to a disjoint union $T \sqcup T$ in $M(P^3(4), \tau_j)$. This means that the two parts obtained by cutting each $(P^3(4), \tau_j)$ horizontally correspond to two $(\mathbb{Z}_2)^3$ -invariant T -handlebodies $T\text{-}HB_{j1}$ and

$T\text{-}HB_{j2}$, each of which is homeomorphic to $T \times [-1, 1]$, as shown in the following figure:



Obviously, all $T\text{-}HB_{j1}$'s are equivariantly homeomorphic to the $T \times [-1, 1]$ with the $(\mathbb{Z}_2)^3$ -action ψ . An easy observation shows that $T\text{-}HB_{j2}, j = 1, 2, 3$, are obtained from the $T \times [-1, 1]$ with the $(\mathbb{Z}_2)^3$ -action ψ by using the following Dehn twists on $T \times [-1, 1]$

$$\begin{aligned} d_1 : (z_1, z_2, t) &\longmapsto (e^{\pi(t+1)\mathbf{i}} z_1, z_2, t) \\ d_2 : (z_1, z_2, t) &\longmapsto (z_1, e^{\pi(t+1)\mathbf{i}} z_2, t) \\ d_3 : (z_1, z_2, t) &\longmapsto (e^{\pi(t+1)\mathbf{i}} z_1, e^{\pi(t+1)\mathbf{i}} z_2, t), \end{aligned}$$

respectively. Namely, the topological types of $T\text{-}HB_{j2} (j = 1, 2, 3)$ are

$$\begin{aligned} d_1(T \times [-1, 1]) &= \{(e^{\pi(t+1)\mathbf{i}} z_1, z_2, t) \mid z_1, z_2 \in S^1, t \in [-1, 1]\} \\ d_2(T \times [-1, 1]) &= \{(z_1, e^{\pi(t+1)\mathbf{i}} z_2, t) \mid z_1, z_2 \in S^1, t \in [-1, 1]\} \\ d_3(T \times [-1, 1]) &= \{(e^{\pi(t+1)\mathbf{i}} z_1, e^{\pi(t+1)\mathbf{i}} z_2, t) \mid z_1, z_2 \in S^1, t \in [-1, 1]\} \end{aligned}$$

respectively, and they admit the $(\mathbb{Z}_2)^3$ -actions which are compatible with the $(\mathbb{Z}_2)^3$ -action ψ on $T \times [-1, 1]$, as follows:

- (i) The $(\mathbb{Z}_2)^3$ -action ψ_1 on $d_1(T \times [-1, 1])$ is given by the following three commutative involutions

$$\begin{aligned} t_1 : (e^{\pi(t+1)\mathbf{i}} z_1, z_2, t) &\longmapsto (e^{\pi(t+1)\mathbf{i}} \bar{z}_1, z_2, t) \\ t_2 : (e^{\pi(t+1)\mathbf{i}} z_1, z_2, t) &\longmapsto (e^{\pi(t+1)\mathbf{i}} z_1, \bar{z}_2, t) \\ t_3 : (e^{\pi(t+1)\mathbf{i}} z_1, z_2, t) &\longmapsto (e^{\pi(t+1)\mathbf{i}} z_1, z_2, -t) \end{aligned}$$

satisfying $\psi d_1 = d_1 \psi_1$.

- (ii) The $(\mathbb{Z}_2)^3$ -action ψ_2 on $d_2(T \times [-1, 1])$ is given by the following three commutative involutions

$$\begin{aligned} t_1 : (z_1, e^{\pi(t+1)\mathbf{i}} z_2, t) &\longmapsto (\bar{z}_1, e^{\pi(t+1)\mathbf{i}} z_2, t) \\ t_2 : (z_1, e^{\pi(t+1)\mathbf{i}} z_2, t) &\longmapsto (z_1, e^{\pi(t+1)\mathbf{i}} \bar{z}_2, t) \\ t_3 : (z_1, e^{\pi(t+1)\mathbf{i}} z_2, t) &\longmapsto (z_1, e^{\pi(t+1)\mathbf{i}} z_2, -t) \end{aligned}$$

satisfying $\psi d_2 = d_2 \psi_2$.

(iii) The $(\mathbb{Z}_2)^3$ -action ψ_3 on $d_3(T \times [-1, 1])$ is given by the following three commutative involutions

$$\begin{aligned} t_1 : (e^{\pi(t+1)\mathbf{i}} z_1, e^{\pi(t+1)\mathbf{i}} z_2, t) &\longmapsto (e^{\pi(t+1)\mathbf{i}} \bar{z}_1, e^{\pi(t+1)\mathbf{i}} z_2, t) \\ t_2 : (e^{\pi(t+1)\mathbf{i}} z_1, e^{\pi(t+1)\mathbf{i}} z_2, t) &\longmapsto (e^{\pi(t+1)\mathbf{i}} z_1, e^{\pi(t+1)\mathbf{i}} \bar{z}_2, t) \\ t_3 : (e^{\pi(t+1)\mathbf{i}} z_1, e^{\pi(t+1)\mathbf{i}} z_2, t) &\longmapsto (e^{\pi(t+1)\mathbf{i}} z_1, e^{\pi(t+1)\mathbf{i}} z_2, -t) \end{aligned}$$

satisfying $\psi d_3 = d_3 \psi_3$.

When $t = \pm 1$, we have $e^{\pi(t+1)\mathbf{i}} = 1$, so we see that each $M(P^3(4), \tau_j)$ is obtained by equivariantly gluing $T \times [-1, 1]$ and $d_j(T \times [-1, 1])$ along their boundaries via the identity of T . On the other hand, when $t = 0$, we have $e^{\pi(t+1)\mathbf{i}} = -1$, so we see that the three Dehn twists d_1, d_2, d_3 determine exactly three monodromy maps $\sigma_j : T \rightarrow T, j = 1, 2, 3$, as follows:

$$\begin{aligned} \sigma_1 : (z_1, z_2) &\longmapsto (z_1, z_2) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = (-z_1, z_2) \\ \sigma_2 : (z_1, z_2) &\longmapsto (z_1, z_2) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = (z_1, -z_2) \\ \sigma_3 : (z_1, z_2) &\longmapsto (z_1, z_2) \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = (-z_1, -z_2). \end{aligned}$$

This completes the proof. \square

Let $(P^3(4), \tau_j), j = 4, 5, 6$, denote those three colored 4-sided prisms shown in Figure (Q) of Section 3. In a similar way, we can prove the following lemma.

Lemma 5.4. *$M(P^3(4), \tau_j), j = 4, 5, 6$, are equivariantly homeomorphic to three twisted K -bundles over S^1 with monodromy maps $f_{-1,1}, f_{1,-1}$ and $f_{-1,-1} \in \mathbf{MCG}^*(K)$ respectively, where the $(\mathbb{Z}_2)^3$ -action on each twisted K -bundle over S^1 is induced by the $(\mathbb{Z}_2)^3$ -action κ on $K \times [-1, 1]$ defined by the following three commutative involutions*

$$\begin{aligned} t_1 : ([z_1, z_2], t) &\longmapsto ([\bar{z}_1, z_2], t) \\ t_2 : ([z_1, z_2], t) &\longmapsto ([z_1, \bar{z}_2], t) \\ t_3 : ([z_1, z_2], t) &\longmapsto ([z_1, z_2], -t). \end{aligned}$$

Let $N = T_0 \cup_{\partial} M_0$ where T_0 is a punctured torus and M_0 is a Möbius band with $T_0 \cap M_0 = \partial T_0 = \partial M_0$. Then N is homeomorphic to $\mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$. It is well known (see [1]) that any diffeomorphism of N is isotopic to one leaving T_0 and M_0 invariant, and there is the following result.

Lemma 5.5 ([1]). *The extended mapping class group $\mathbf{MCG}_+^*(N)$ of N is isomorphic to $\mathrm{GL}(2, \mathbb{Z})$, and the isomorphism is given by the natural homomorphism*

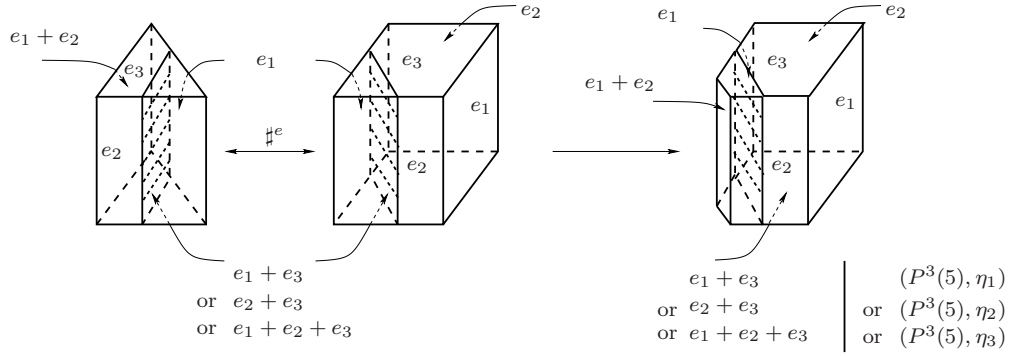
$$\Pi : \mathbf{MCG}_+^*(N) \rightarrow \mathrm{Aut}(H_1(N; \mathbb{Z}) / \mathrm{Tor}(H_1(N; \mathbb{Z}))) = \mathrm{Aut}(H_1(T; \mathbb{Z})) \cong \mathrm{GL}(2, \mathbb{Z})$$

where $T = T_0 \cup_{\partial} D^2$ is a torus.

Let $(P^3(5), \eta_j), j = 1, 2, 3$, denote those three colored 5-sided prisms shown in Figure (R) of Section 3. Then we have

Lemma 5.6. $M(P^3(5), \eta_j), j = 1, 2, 3$, are equivariantly homeomorphic to three special twisted N -bundles over S^1 with monodromy maps as the inverse images of $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \text{GL}(2, \mathbb{Z})$ respectively under the isomorphism Π .

Proof. In fact, each $(P^3(5), \eta_j)$ can be constructed by using a colored 3-sided prism and a colored 4-sided prism under the operation \sharp^e , as shown in the following figure:



By Lemmas 4.2 and 4.3, each colored 3-sided prism used above corresponds to a trivial $\mathbb{R}P^2$ -bundle over S^1 , and by Lemma 5.3 the three colored 4-sided prisms used above correspond to the three nontrivial T -bundles over S^1 with monodromy matrices $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, respectively. So each $M(P^3(5), \eta_j)$ is equivariantly homeomorphic to a non-trivial N -bundle over S^1 with the desired monodromy map. \square

Now let us look at how the operation \sharp^\odot on \mathcal{M} works. To give a statement in detail, we divide our discussion into the following three cases.

- (1) If we exactly cut out a 2-independent triangular facet from (P^3, λ) , then we also need to cut out such a facet from a colored 3-sided prism $(P^3(3), \tau)$. According to the colorings on $P^3(3)$, the topological type of $M(P^3(3), \tau)$ must be $S^1 \times \mathbb{R}P^2$, so we can cut out a $(\mathbb{Z}_2)^3$ -invariant $\mathbb{R}P^2 \times (-1, 1)$ from $S^1 \times \mathbb{R}P^2$ with a certain action ϕ . Then we glue $M(P^3, \lambda) \setminus (\mathbb{R}P^2 \times (-1, 1))$ and $(S^1 \times \mathbb{R}P^2, \phi) \setminus (\mathbb{R}P^2 \times (-1, 1))$ along their boundaries, i.e.,

$$\begin{aligned}
 & M(P^3, \lambda) \sharp^\odot M(P^3(3), \tau) = M(P^3, \lambda) \sharp^\odot (S^1 \times \mathbb{R}P^2, \phi) \\
 & = (M(P^3, \lambda) \setminus (\mathbb{R}P^2 \times (-1, 1))) \cup_{\mathbb{R}P^2 \sqcup \mathbb{R}P^2} ((S^1 \times \mathbb{R}P^2, \phi) \setminus (\mathbb{R}P^2 \times (-1, 1))).
 \end{aligned}$$

- (2) If we exactly cut out a 2-independent square facet F from (P^3, λ) , then we need a colored 4-sided prism $(P^3(4), \tau)$ to do a coloring change of F . In this case, the section in (P^3, λ) or $(P^3(4), \tau)$ is a 2-independent square section S_\square . If S_\square is 2-colorable (i.e., S_\square corresponds to a disjoint union $T \sqcup T$ by Lemma 5.1), then by

Lemma 5.3 $M(P^3(4), \tau)$ is equivariantly homeomorphic to a twisted T -bundle over S^1 , and we can cut out a $(\mathbb{Z}_2)^3$ -invariant $T \times (-1, 1)$ from $M(P^3(4), \tau)$. If S_\square is 3-colorable (i.e., S_\square corresponds to a disjoint union $K \sqcup K$ by Lemma 5.1), then by Lemma 5.3, $M(P^3(4), \tau)$ is equivariantly homeomorphic to a twisted K -bundle over S^1 , and we can cut out a $(\mathbb{Z}_2)^3$ -invariant $K \times (-1, 1)$ from $M(P^3(4), \tau)$. Combining these arguments, we conclude that if the topological type of $M(P^3(4), \tau)$ is a twisted T -bundle over S^1 , then

$$\begin{aligned} & M(P^3, \lambda) \#^{\widetilde{\odot}} M(P^3(4), \tau) \\ &= (M(P^3, \lambda) \setminus (T \times (-1, 1))) \cup_{T \sqcup T} (M(P^3(4), \tau) \setminus (T \times (-1, 1))) \end{aligned}$$

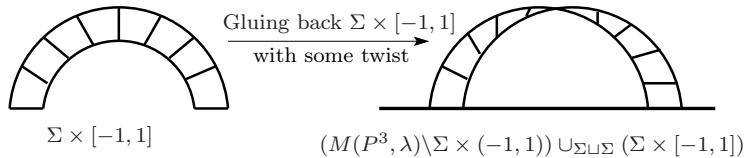
and if the topological type of $M(P^3(4), \tau)$ is a twisted K -bundle over S^1 , then

$$\begin{aligned} & M(P^3, \lambda) \#^{\widetilde{\odot}} M(P^3(4), \tau) \\ &= (M(P^3, \lambda) \setminus (K \times (-1, 1))) \cup_{K \sqcup K} (M(P^3(4), \tau) \setminus (K \times (-1, 1))). \end{aligned}$$

- (3) If we exactly cut out a 2-independent pentagonal facet F from (P^3, λ) , then we need a colored 5-sided prism $(P^3(5), \tau)$ to change the coloring of F . Since the section of (P^3, λ) or $(P^3(5), \tau)$ is a 2-independent pentagonal section $S_{\mathbf{x}}$, by Lemmas 5.1 and 5.6, $M(P^3(5), \tau)$ is equivariantly homeomorphic to a twisted N -bundle over S^1 , and we can cut out a $(\mathbb{Z}_2)^3$ -invariant $N \times (-1, 1)$ from $M(P^3(5), \tau)$. Then the operation $\#^{\widetilde{\odot}}$ of $M(P^3, \lambda)$ and $M(P^3(5), \tau)$ is as follows:

$$\begin{aligned} & M(P^3, \lambda) \#^{\widetilde{\odot}} M(P^3(5), \tau) \\ &= (M(P^3, \lambda) \setminus (N \times (-1, 1))) \cup_{N \sqcup N} (M(P^3(5), \tau) \setminus (N \times (-1, 1))). \end{aligned}$$

Remark 5.2. In doing the operation $\#^{\widetilde{\odot}}$ on a $M(P^3, \lambda)$, we cut out a small facet from (P^3, λ) and a bottom facet from a colored i -sided prism $(P^3(i), \tau)$, $i = 3, 4, 5$, and then glue them together along their sections. There are similar procedures for $M(P^3, \lambda)$ and $M(P^3(i), \tau)$. Namely, we first remove an open $(\mathbb{Z}_2)^3$ -invariant Σ -handlebody $\Sigma \times (-1, 1)$ from $M(P^3, \lambda)$ and $M(P^3(i), \tau)$ respectively where Σ is a $\mathbb{R}P^2$, or a torus, or a Klein bottle, or a $\mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$, and then glue back the remaining part (i.e., a $(\mathbb{Z}_2)^3$ -invariant Σ -handlebody $\Sigma \times [-1, 1]$) of $M(P^3(i), \tau)$ to $M(P^3, \lambda) \setminus \Sigma \times (-1, 1)$ along their boundaries. When $i = 3$, $M(P^3(3), \tau)$ is a $\mathbb{R}P^2$ -bundle over S^1 but it is always trivial, so we can glue back the remaining part of $M(P^3(3), \tau)$ to $M(P^3, \lambda) \setminus \Sigma \times (-1, 1)$ without any twist. However, when $i = 4$ or 5 , since $M(P^3(i), \tau)$ is always a non-trivial bundle over S^1 by Lemmas 5.3, 5.4 and 5.6, this means that gluing back $\Sigma \times [-1, 1]$ actually leads to the appearance of some twist of $\Sigma \times [-1, 1]$, as shown in the following figure:



Remark 5.3. When we do the operations $\widetilde{\sharp}$ and $\widetilde{\sharp}^{\odot}$ on \mathcal{M} , we see that after removing an open $(\mathbb{Z}_2)^3$ -invariant desired 3-manifold from $M(\odot, \tau)$ or $M(P^3(i), \tau)$ ($i = 3, 4, 5$), the remaining part is still a same type of $(\mathbb{Z}_2)^3$ -invariant 3-manifold with boundary but admits a different $(\mathbb{Z}_2)^3$ -action. Of course, the actions on these two 3-manifolds are compatible with the action on $M(\odot, \tau)$ or $M(P^3(i), \tau)$ ($i = 3, 4, 5$). This means that $M(\odot, \tau)$ and $M(P^3(i), \tau)$ ($i = 3, 4, 5$) admit equivariant Heegaard splittings (cf [7]).

6. APPLICATION TO EQUIVARIANT COBORDISM

Stong showed in [18] that the $(\mathbb{Z}_2)^n$ -equivariant unoriented cobordism class of each closed $(\mathbb{Z}_2)^n$ -manifold is determined by that of its fixed data. This gives the following result in the special case.

Proposition 6.1 (Stong). *Suppose that a closed manifold M^n admits a $(\mathbb{Z}_2)^n$ -action such that its fixed point set is finite. Then M^n bounds equivariantly if and only if the tangent representations at fixed points appear in pairs up to isomorphism.*

Each n -dimensional small cover $\pi : M^n \longrightarrow P^n$ has a finite fixed point set, which just corresponds to the vertex set of P^n . By GKM theory [6], its tangent representations at fixed points exactly correspond to a $\text{Hom}((\mathbb{Z}_2)^n, \mathbb{Z}_2)$ -coloring on the 1-skeleton of P^n . It is not difficult to check that this $\text{Hom}((\mathbb{Z}_2)^n, \mathbb{Z}_2)$ -coloring on the 1-skeleton of P^n is algebraically dual to the $(\mathbb{Z}_2)^n$ -coloring on P^n , as seen in the proof of Lemma 4.3. Therefore, we have that the $(\mathbb{Z}_2)^n$ -colorings of two vertices v_1, v_2 in P^n are the same if and only if the corresponding tangent representations at the two fixed points $\pi^{-1}(v_1), \pi^{-1}(v_2)$ are isomorphic. Moreover, by Proposition 6.1 we conclude that

Corollary 6.2. *Let $\pi : M^n \longrightarrow P^n$ be a small cover over P^n . Then the $(\mathbb{Z}_2)^n$ -colorings of all vertices in P^n appear in pairs if and only if M^n bounds equivariantly.*

Now let us look at how six operations work in $\widehat{\mathcal{M}}$. Given two classes $[M(P_1^3, \lambda_1)]$ and $[M(P_2^3, \lambda_2)]$ in $\widehat{\mathcal{M}}$, when we do the operation $\widetilde{\sharp}^v$ on $M(P_1^3, \lambda_1)$ and $M(P_2^3, \lambda_2)$, we need to cut out two vertices with same coloring from (P_1^3, λ_1) and (P_2^3, λ_2) respectively. This means that we exactly cancel two fixed points with same tangent representation in $M(P_1^3, \lambda_1) \sqcup M(P_2^3, \lambda_2)$, but this does not change $M(P_1^3, \lambda_1) \sqcup M(P_2^3, \lambda_2)$ up to equivariant cobordism by Proposition 6.1. Thus we have

Lemma 6.3. *Let $[M(P_1^3, \lambda_1)]$ and $[M(P_2^3, \lambda_2)]$ be two classes in $\widehat{\mathcal{M}}$. Then*

$$[M(P_1^3, \lambda_1) \widetilde{\sharp}^v M(P_2^3, \lambda_2)] = [M(P_1^3, \lambda_1)] + [M(P_2^3, \lambda_2)].$$

By a similar argument, we have

Lemma 6.4. *Let $[M(P^3, \lambda)]$ be a class in $\widehat{\mathcal{M}}$. Then*

$$\begin{aligned} [M(P^3, \lambda) \#^e M(P^3(3), \tau)] &= [M(P^3, \lambda)] + [M(P^3(3), \tau)] \\ [M(P^3, \lambda) \#^{\widetilde{eve}} M(P^3(3), \tau)] &= [M(P^3, \lambda)] + [M(P^3(3), \tau)] \\ [M(P^3, \lambda) \#^{\widetilde{q}} M(\emptyset, \tau)] &= [M(P^3, \lambda)] \\ [M(P^3, \lambda) \#^{\widetilde{\odot}} M(P^3(i), \tau)] &= [M(P^3, \lambda)] + [M(P^3(i), \tau)], i = 3, 4, 5. \end{aligned}$$

Remark 6.1. Lemmas 6.3 and 6.4 tell us that five operations $\#^v, \#^e, \#^{\widetilde{eve}}, \#^{\widetilde{q}}, \#^{\widetilde{\odot}}$ have a nice compatibility with the disjoint union in the sense of equivariant cobordism. Notice that clearly $M(\emptyset, \tau)$ bounds equivariantly by Proposition 6.1, so $[M(\emptyset, \tau)] = 0$ in $\widehat{\mathcal{M}}$.

However, the operation $\#^{\widetilde{\Delta}}$ is a little different from other five operations in $\widehat{\mathcal{M}}$. Let $[M(P_1^3, \lambda_1)]$ and $[M(P_2^3, \lambda_2)]$ be two classes in $\widehat{\mathcal{M}}$. When we do the operation $\#^{\widetilde{\Delta}}$ on $M(P_1^3, \lambda_1)$ and $M(P_2^3, \lambda_2)$, it is possible that we just cut out two triangular facets with different colorings from (P_1^3, λ_1) and (P_2^3, λ_2) respectively. If this happens, then we glue the two parts cut out from (P_1^3, λ_1) and (P_2^3, λ_2) along their sections, so that we can form a 3-sided prism $P^3(3)$ with a natural induced coloring (denoted by $\lambda_1 \#^{\Delta} \lambda_2$) such that top and bottom facets are colored differently. Furthermore, this colored 3-sided prism can be recovered into a small cover. Thus, by Proposition 6.1 we have

Lemma 6.5. *Let $[M(P_1^3, \lambda_1)]$ and $[M(P_2^3, \lambda_2)]$ be two classes in $\widehat{\mathcal{M}}$. Then*

$$\begin{aligned} &[M(P_1^3, \lambda_1) \#^{\widetilde{\Delta}} M(P_2^3, \lambda_2)] \\ &= \begin{cases} [M(P_1^3, \lambda_1)] + [M(P_2^3, \lambda_2)] & \text{if we cut out two triangular} \\ & \text{facets with same coloring} \\ [M(P_1^3, \lambda_1)] + [M(P_2^3, \lambda_2)] + [M(P^3(3), \lambda_1 \#^{\Delta} \lambda_2)] & \text{if we cut out two triangular} \\ & \text{facets with different colorings.} \end{cases} \end{aligned}$$

Finally, Theorem 1.3 follows immediately from Theorem 1.2 and Lemmas 6.3, 6.4 and 6.5.

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